Physics of the Atmosphere

Lecture 12: Conservation Laws in the Atmosphere (II)

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Conservation Laws

- Why conservation laws?
- A system of governing equations includes the conservation of mass (air and water vapor, separately), momentum and energy.
- The governing equations of a mathematical model describe how the values of the unknown variables (i.e., the dependent variables) change when one or more of the known (i.e., independent) variables change.

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Conservation Laws

- **Governing equations are based on physical principles derived from the conservation of**
  - Mass ➔ (Mass) Continuity Equ. ➔ ρ
  - Momentum ➔ Equ. of Motion (or Momentum Equ.) ➔ $\mathbf{v}$=(u,v,w)
  - Energy (1st Law of Thermodynamics) ➔ Thermodynamic Equ. (or Temperature Equ.) ➔ T or θ
  - Water Mass ➔ Water Vapor Budget (Moisture) Equ. ➔ $q_v, \cdots, q_h$
  - Ideal Gas Law ➔ Equ. of State (General Gas Equ.) ➔ p

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Conservation of Mass

- **We exclude water vapor, for the moment. ← Phase change**
- **Mass is neither created nor destroyed.**
- **The density comes in here instead of mass (mass per volume):**
  - If the density at a location increases, then the density has to be transferred from some other place to this location.
  - Thus, the total change of density in a moving air parcel is proportional to the divergence of the air stream in the volume.

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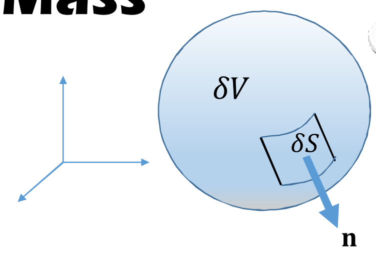
Conservation of Mass

- **Lagrangian view:**



$$\frac{DM}{Dt} \equiv 0 = \frac{D(\rho V)}{Dt}$$

$$\frac{D\rho}{Dt} + \frac{\rho}{V}\frac{DV}{Dt} = \boxed{\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0}$$

Continuity Equation
(velocity divergence form)

$$\iiint_V \nabla \cdot \mathbf{Q}\, dV = \iint_S \mathbf{Q} \cdot \mathbf{n}\, dS = \iint_S \mathbf{Q} \cdot d\mathbf{s}$$

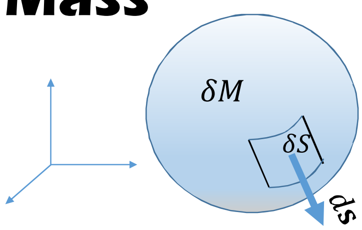
← Divergence theorem

Practice: Using the divergence theorem, show that

$$\frac{1}{V}\frac{DV}{Dt} = \nabla \cdot \mathbf{v}$$

Conservation of Mass

- **Lagrangian view**
  - Considers the mass δM moving together with the air flux at the internal of a material surface δS



$$\frac{DM}{Dt} \equiv 0 = \frac{D(\rho V)}{Dt}$$

$$\frac{D\rho}{Dt} + \frac{\rho}{V}\frac{DV}{Dt} = \boxed{\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0}$$

Continuity Equation

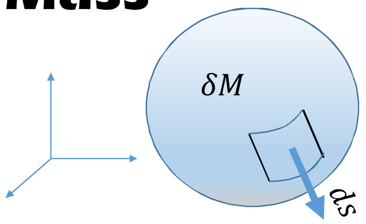
$$\frac{D\delta M}{Dt} \equiv 0 = \frac{D(\rho \delta V)}{Dt} = \delta V \frac{D\rho}{Dt} + \rho \frac{D\delta V}{Dt} \quad (1)$$

Practice: From (1), derive the continuity equation.

$$\delta V \frac{D\rho}{Dt} + \rho \boxed{\frac{D\delta V}{Dt}} = 0 = \delta V \frac{D\rho}{Dt} + \boxed{\delta V \rho (\nabla \cdot \mathbf{v})} \Rightarrow \frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0$$

Conservation of Mass

- **Lagrangian view:**



$$\frac{DM}{Dt} \equiv 0 = \frac{D(\rho V)}{Dt}$$

$$\frac{D\delta M}{Dt} \equiv 0 = \frac{D(\rho\delta V)}{Dt} = \delta V\frac{D\rho}{Dt} + \rho\frac{D\delta V}{Dt}$$

$$\frac{D\delta V}{Dt} = \frac{D}{Dt}(\delta x\delta y\delta z) = \delta y\delta z\frac{D\delta x}{Dt} + \delta x\delta z\frac{D\delta y}{Dt} + \delta x\delta y\frac{D\delta z}{Dt}$$

$$\frac{1}{\delta q}\frac{D\delta q}{Dt} = \frac{\partial}{\partial q}\frac{Dq}{Dt}$$

$$\frac{D\rho}{Dt} + \frac{\rho}{V}\frac{DV}{Dt} = \boxed{\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{v} = 0}$$

Continuity Equation

$$\frac{D\delta V}{Dt} = \delta x\delta y\delta z\left(\left(\frac{\partial}{\partial x}\frac{Dx}{Dt}\right) + \left(\frac{\partial}{\partial y}\frac{Dy}{Dt}\right) + \left(\frac{\partial}{\partial z}\frac{Dz}{Dt}\right)\right) = \delta V\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$\delta V\frac{D\rho}{Dt} + \rho\frac{D\delta V}{Dt} = 0 = \delta V\frac{D\rho}{Dt} + \delta V\rho(\nabla\cdot\mathbf{v}) \Longrightarrow \frac{D\rho}{Dt} + \rho(\nabla\cdot\mathbf{v}) = 0$$

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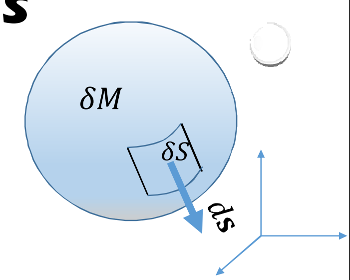
Conservation of Mass

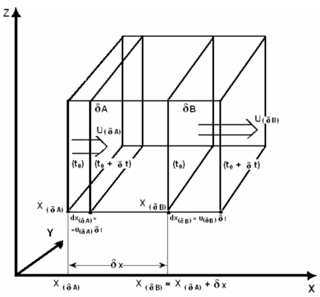
- **Lagrangian view: Summary**
  - Consider a mass δM, of volume $\delta V = \delta x\delta y\delta z$, moving (with δM constant) with the flow
  - All particles in δV enclosed by δS never abandon δV remaining forever in δS by definition
  - Thus, as $\delta M = \rho\delta V$ is conserved during the motion (despite both ρ and δV can individually vary):



$$\frac{1}{\delta M}\frac{D(\delta M)}{Dt} = \frac{1}{\rho\delta V}\frac{D(\rho\delta V)}{Dt} = \frac{1}{\rho\delta V}\left[\rho\frac{D(\delta V)}{Dt} + \delta V\frac{D\rho}{Dt}\right] = \frac{1}{\rho}\frac{D\rho}{Dt} + \frac{1}{\delta V}\frac{D(\delta V)}{Dt}$$
$$= \frac{1}{\rho}\frac{D\rho}{Dt} + \frac{1}{\delta x}\frac{D(\delta x)}{Dt} + \frac{1}{\delta y}\frac{D(\delta y)}{Dt} + \frac{1}{\delta z}\frac{D(\delta z)}{Dt}$$

  - Being $\delta u = D(\delta x)/Dt$ and so on, and taking the limit for δV ➔0:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{D\rho}{Dt} = \nabla\bullet\mathbf{v} + \frac{1}{\rho}\frac{D\rho}{Dt} = 0$$



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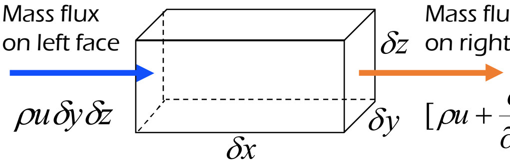
Conservation of Mass

- **Eulerian view:**
  - Considers the entering and leaving of mass fluxes through the surfaces of a cube hose wall are permeable to the air passage



$\rho u \delta y \delta z$ $\quad \delta x \quad \delta y \quad \delta z \quad$ $[\rho u + \frac{\partial}{\partial x}(\rho u)\delta x]\delta y \delta z$

Net increase of mass per unit volume in x-direction: $-\frac{\partial}{\partial x}(\rho u)$

Net increase of mass per unit volume in y-direction: $-\frac{\partial}{\partial y}(\rho v)$

Net increase of mass per unit volume in z-direction: $-\frac{\partial}{\partial z}(\rho w)$

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Conservation of Mass

- **Eulerian view:**

Thus, the net increase of mass per unit volume

$$= -\left(\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right) = -\nabla \bullet (\rho \mathbf{v})$$

= rate of change of density $= \frac{\partial \rho}{\partial t}$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{v}) = 0}$$

Continuity Equation

(mass divergence form)

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Conservation of Mass

- **Eulerian view: Summary**
  - Mass flux: $\mathbf{j} = \rho\mathbf{v}$
  - Mass balance of a fluid volume dV with mass dM=rdV:

$$\frac{\partial}{\partial t}\int_V \rho dV = -\oint_S \mathbf{j}\cdot d\mathbf{s} = -\oint_S \rho\mathbf{v}\cdot d\mathbf{s}$$

  - Exchanging time derivative and integral on the left side and applying the divergence theorem to the right side yields:
  - From which: $\int_V \frac{\partial\rho}{\partial t}dV = -\int_V \nabla\bullet(\rho\mathbf{v})dV \quad \Rightarrow \quad \int_V \left[\frac{\partial\rho}{\partial t} + \nabla\bullet(\rho\mathbf{v})\right]dV = 0$

$$\frac{\partial\rho}{\partial t} + \nabla\bullet(\rho\mathbf{v}) = 0$$

Conservation of Mass

- **(Mass) Continuity Equation:**

$$\frac{\partial\rho}{\partial t} + \nabla\bullet(\rho\mathbf{v}) = \overbrace{\frac{\partial\rho}{\partial t} + \mathbf{v}\bullet\nabla\rho}^{\equiv \frac{D\rho}{Dt}} + \rho\nabla\bullet\mathbf{v} = 0$$

$$\Rightarrow \frac{D\rho}{Dt} + \rho\nabla\bullet\mathbf{v} = 0$$

$$\Rightarrow \frac{1}{\rho}\frac{D\rho}{Dt} + \nabla\bullet\mathbf{v} = 0$$

$$\Rightarrow \frac{D}{Dt}(\ln\rho) + \nabla\bullet\mathbf{v} = 0$$

Conservation of Mass

- **(Mass) Continuity Equation:**
  - Horizontal pressure and density gradients are usually very small (but not in the vertical, see hydrostatic equation)
  - In case of an incompressible fluid, we have $\partial\rho/\partial t = 0$ and $\nabla\rho = 0$; thus, from the continuity equation: $\nabla \cdot \mathbf{v} = 0$
    - That is, the velocity field is divergence-free.
  - In this case, a horizontal divergence or convergence ($\nabla_H = \partial u/\partial x + \partial v/\partial y$) is usually coincident with vertical air movement ($\partial w/\partial z$).

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Divergence vs. Advection

- **Divergence:** $\nabla \bullet \mathbf{v}$

$$\nabla \bullet \mathbf{v} = \left(\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}\right) \bullet \left(u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + w\hat{\mathbf{k}}\right)$$

$$= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \qquad \Rightarrow \qquad \nabla \bullet (\rho \mathbf{v}) = \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}$$

- **Advection:** $\mathbf{v} \bullet \nabla$

$$\mathbf{v} \bullet \nabla = \left(u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + w\hat{\mathbf{k}}\right) \bullet \left(\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}\right)$$

$$= \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right) \qquad \Rightarrow \qquad \mathbf{v} \bullet \nabla\rho = u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}$$

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Conservation of Momentum

- **Newton's Laws of Motion**
  - The momentum of a body remains constant unless the body is acted upon by a net force ➔ Conservation of Momentum
  - The rate of change of momentum of a body is proportional to the net force acting on the body and is in the same direction of the net force ➔ $\mathbf{F} = m\mathbf{a}$
  - For every net force acting on a body, there is a corresponding force of the same magnitude exerted by the body in the opposite direction.

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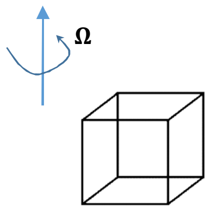
Conservation of Momentum

- **Newton's Laws of Motion**

$$\mathbf{F} = m\mathbf{a} \Rightarrow \frac{\mathbf{F}}{m} = \mathbf{a}$$

$$\mathbf{a}_{\text{net}} = \frac{\mathbf{F}_{\text{net}}}{m} = \frac{\sum_i F_i}{m} =$$



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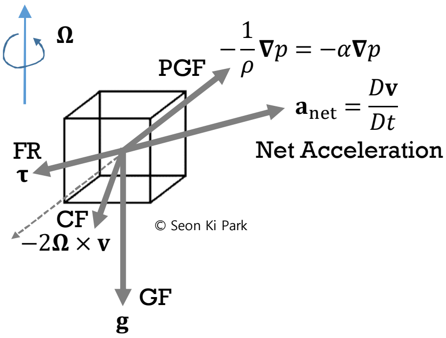
Conservation of Momentum

- Newton's Laws of Motion

$$\mathbf{F} = m\mathbf{a} \Rightarrow \frac{\mathbf{F}}{m} = \mathbf{a}$$

$$\mathbf{a}_{\text{net}} = \frac{\mathbf{F}_{\text{net}}}{m} = \frac{\sum_i F_i}{m} = \mathbf{a}_{\text{PGF}} + \mathbf{a}_{\text{CF}} + \mathbf{a}_{\text{GF}} + \mathbf{a}_{\text{FR}}$$



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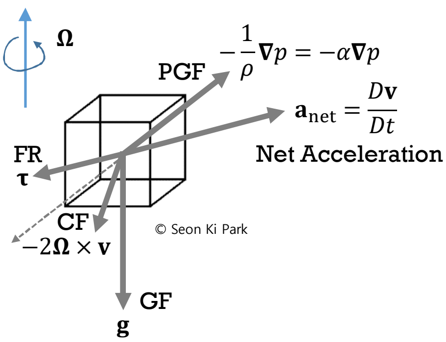
Conservation of Momentum

- Newton's Laws of Motion

$$\mathbf{F} = m\mathbf{a} \Rightarrow \frac{\mathbf{F}}{m} = \mathbf{a}$$



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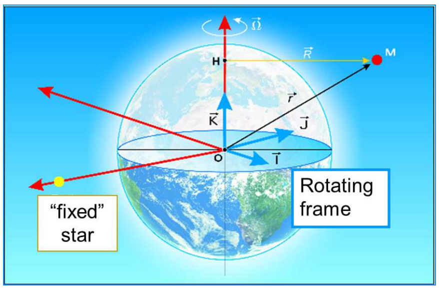
Conservation of Momentum

- Newton's Laws of Motion ➔ Equ. of Motion (Momentum Equ.)

$$\boxed{\frac{D\mathbf{v}}{Dt} = \underbrace{-2\boldsymbol{\Omega}\times\mathbf{v}}_{\text{Coriolis acceleration}} \ \underbrace{-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})}_{\text{centrifugal acceleration}} + \underbrace{\mathbf{F}/m}_{\text{acceleration by "absolute" forces}}}$$



(Malardel, 2010; ECMWF Lecture)

$$\frac{\mathbf{F}}{m} = \underbrace{-\alpha\nabla p}_{\substack{\text{acceleration by}\\ \text{press. grad. force}}} + \underbrace{\mathbf{g}^*}_{\substack{\text{acceleration by}\\ \text{gravitational force}}} + \underbrace{\boldsymbol{\tau}}_{\substack{\text{acceleration by}\\ \text{frictional force}}} \qquad (\alpha = 1/\rho)$$

Conservation of Momentum

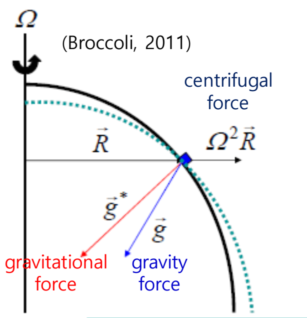
- Newton's Laws of Motion ➔ Equ. of Motion (Momentum Equ.)

$$\frac{\mathbf{F}}{m} = \underbrace{-\alpha\nabla p}_{\substack{\text{acceleration by}\\ \text{press. grad. force}}} + \underbrace{\mathbf{g}^*}_{\substack{\text{acceleration by}\\ \text{gravitational force}}} + \underbrace{\boldsymbol{\tau}}_{\substack{\text{acceleration by}\\ \text{frictional force}}} \qquad (\alpha = 1/\rho)$$

$$\frac{D\mathbf{v}}{Dt} = -\alpha\nabla p - 2\boldsymbol{\Omega}\times\mathbf{v} \ \underbrace{-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}) + \mathbf{g}^*}_{-g\hat{\mathbf{k}}:\ \text{gravity accel.}} + \boldsymbol{\tau}$$

$$\boxed{\frac{D\mathbf{v}}{Dt} = -\alpha\nabla p - 2\boldsymbol{\Omega}\times\mathbf{v} - g\hat{\mathbf{k}} + \boldsymbol{\tau}}$$

Equation of Motion or Momentum Equation



The earth has adjusted to this balance of forces by taking the shape of a spheroid with an equatorial bulge.

(Malardel, 2010; ECMWF Lecture)

Conservation of Energy

- **1st Law of Thermodynamics**
  - Energy is neither created nor destroyed. ➔ Conservation of Energy
  - Change in internal energy (U) is due to a combination of heat (Q) added to the system and work done by the system (W).

$$\Delta U = \Delta Q - \Delta W$$

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Conservation of Energy

- **1st Law of Thermodynamics**

rate of work done by gas on its surroundings by compression/expansion

$$\frac{dU}{dt} = \dot{Q} - \dot{W}$$

rate of heat exchange with the surroundings

For a perfect gas, $U = c_v T$ (c_v = specific heat at constant volume),

$$\dot{W} = p\frac{d\alpha}{dt} \qquad (\alpha = 1/\rho)$$

$$\boxed{c_v \frac{dT}{dt} + p\frac{d\alpha}{dt} = \dot{Q}}$$

diabatic heating rate

rate of change of IE

conversion between thermal and mechanical E

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Conservation of Energy

- **Thermodynamic Equation:**

$$c_v \frac{dT}{dt} + p\frac{d\alpha}{dt} = \dot{Q}$$

$$c_p \frac{dT}{dt} - \alpha\frac{dp}{dt} = \dot{Q}$$

$\dot{Q} = 0$ for adiabatic process

Conservation of Energy

- **Thermodynamic Equation:**

$$c_p \frac{dT}{dt} - \alpha\frac{dp}{dt} = \dot{Q}$$

$$c_p \frac{dT}{dt} = \dot{Q} + \alpha\left(\frac{\partial p}{\partial t} + \mathbf{v}_H \bullet \nabla p + w\frac{\partial p}{\partial z}\right) = \dot{Q} + \alpha\left(\overset{\text{small}}{\cancel{\frac{\partial p}{\partial t}}} + \overset{\text{small}}{\cancel{\mathbf{v}_H \bullet \nabla p}}\right) - gw$$

$$\frac{dT}{dt} = \frac{\dot{Q}}{c_p} - \frac{g}{c_p}w = \frac{\partial T}{\partial t} + \mathbf{v}_H \bullet \nabla T + w\frac{\partial T}{\partial z}$$

$\Gamma_d = \frac{g}{c_p}$; $\Gamma = -\frac{\partial T}{\partial z}$

$$\frac{\partial T}{\partial t} = \frac{\dot{Q}}{c_p} - \left(\frac{g}{c_p} + \frac{\partial T}{\partial z}\right)w - \mathbf{v}_H \bullet \nabla T = \frac{\dot{Q}}{c_p} - (\Gamma_d - \Gamma)w - \mathbf{v}_H \bullet \nabla T$$

(Broccoli, 2011)

Conservation of Energy

- **Thermodynamic Equation:**

$$c_p \frac{dT}{dt} - \alpha \frac{dp}{dt} = \dot{Q}$$

$$\frac{\partial T}{\partial t} = \underbrace{\dot{Q}/c_p}_{A} \underbrace{-(\Gamma_d - \Gamma)w}_{B} \underbrace{-\mathbf{v}_H \bullet \nabla T}_{C}$$

- A: Diabatic Heating
  1. Sensible heating
  2. Latent heating (phase change, i.e., evaporation, condensation)
  3. Radiative heating

- B: Adiabatic Effects
  1. $(\Gamma_d - \Gamma)$ is a measure of stability.
  2. Upward motion in a stable atmosphere is a cooling process.

- C: Horizontal Advection Term
  1. Often this is the largest term

(Broccoli, 2011)

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