Tendency equation

Textbooks and web sites references for this lecture:

- James R. Holton, An Introduction to Dynamic Meteorology, Academic Press, 1992, ISBN 0-12-354355-X (§ 6.4)

Quasi-geostrophic prediction

Characteristics of geostrophic circulation forced by vertical motions associated with vorticity and thermal advection can be determined **without explicitly determining ω**.

Since **T, ζ_g and v_g are all functions of Φ:**

$$\frac{\partial \Phi}{\partial p} = -\alpha = -\frac{RT}{p} \quad \Rightarrow \quad T = -\frac{p}{R}\frac{\partial \Phi}{\partial p}$$

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f_0}\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right) = \frac{\nabla^2 \Phi}{f_0}$$

$$\overrightarrow{v_g} = \frac{\vec{k} \times \nabla \Phi}{f_0}$$

quasi-geostrophic and thermodynamic energy equations can be written so that **they contain only one of two variables: Φ and ω**.

$$\frac{\partial \zeta_g}{\partial t} + \overrightarrow{v_g} \cdot \nabla\left(\zeta_g + f\right) = f_0 \frac{\partial \omega}{\partial p} \qquad \left(\frac{\partial}{\partial t} + \overrightarrow{v_g} \cdot \nabla\right) T - \left(\frac{\sigma p}{R}\right)\omega = \frac{J}{c_p}$$

Quasi-geostrophic system

Lets define geopotential tendency $\chi \equiv \partial\Phi/\partial t$: equations can be rewritten as:

$$\left(\frac{\partial}{\partial t}+\vec{v}_g\cdot\nabla\right)T-\left(\frac{\sigma p}{R}\right)\omega=\frac{J}{c_p}$$

(J=0) ➔

No diabatic heating

$$\frac{\partial\chi}{\partial p}=-\vec{v}_g\cdot\nabla\left(\frac{\partial\Phi}{\partial p}\right)-\sigma\omega$$

$$\frac{\partial\zeta_g}{\partial t}=-\vec{v}_g\cdot\nabla\left(\zeta_g+f\right)+f_0\frac{\partial\omega}{\partial p}$$

➔

$$\nabla^2\chi=-f_0\vec{v}_g\cdot\nabla\left(\frac{1}{f_0}\nabla^2\Phi+f\right)+f_0^2\frac{\partial\omega}{\partial p}$$

These two equations form the **quasi-geostrophic system**.

First one states that vertical derivative of geopotential tendency is equal to sum of thickness advection and adiabatic thickness change owing to vertical motion.

Second one indicates that horizontal laplaciam of geopotential tendency is equal to sum of vorticity advection plus vorticity generation by divergence effect. Purely geostrophic motion (ω=0) is a solution only in very special situations like barotropic flows (no pressure dependence) or zonally symmetric flow (no x dependence).

Geopotential tendency

Multiplying by f_0^2/σ first equation, differentiating with respect to pressure and summing second equation:

$$\frac{f_0^{\,2}}{\sigma}\times\frac{\partial}{\partial p}\left\{\frac{\partial\chi}{\partial p}=-\overrightarrow{v_g}\cdot\nabla\left(\frac{\partial\Phi}{\partial p}\right)-\sigma\omega\right\}\Rightarrow\frac{f_0^{\,2}}{\sigma}\frac{\partial^2\chi}{\partial p^2}=-\frac{f_0^{\,2}}{\sigma}\frac{\partial}{\partial p}\left[\overrightarrow{v_g}\cdot\nabla\left(\frac{\partial\Phi}{\partial p}\right)\right]-f_0^{\,2}\frac{\partial\omega}{\partial p}$$

$$\left\{\nabla^2\chi=-f_0\overrightarrow{v_g}\cdot\nabla\left(\frac{1}{f_0}\nabla^2\Phi+f\right)+f_0^{\,2}\frac{\partial\omega}{\partial p}\right\}$$

and rearranging some terms, we arrive to:

$$\left[\nabla^2+\frac{\partial}{\partial p}\left(\frac{f_0^{\,2}}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi=-f_0\overrightarrow{v_g}\cdot\nabla\left(\frac{1}{f_0}\nabla^2\Phi+f\right)-\frac{\partial}{\partial p}\left[-\frac{f_0^{\,2}}{\sigma}\overrightarrow{v_g}\cdot\nabla\left(-\frac{\partial\Phi}{\partial p}\right)\right]$$

A B C

known as **geopotential tendency equation**. It provides a relation between the local change of geopotential (term A) and the distribution of vorticity (B) and thickness advection (C)

$$\left[\nabla^2+\frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi=-f_0\overrightarrow{v_g}\cdot\nabla\left(\frac{1}{f_0}\nabla^2\Phi+f\right)-\frac{\partial}{\partial p}\left[-\frac{f_0^2}{\sigma}\overrightarrow{v_g}\cdot\nabla\left(-\frac{\partial\Phi}{\partial p}\right)\right]$$

A B C

Letting a similar (sinusoidal) relationship for χ like that for Φ:

$$\chi(x,y,p,t)=X(p,t)\sin kx\cos ly$$

in order that: $$\nabla^2\chi\approx-\left(k^2+l^2\right)\chi$$

and hypothesizing the following dependencies of forcing terms B and C:

$$-f_0\overrightarrow{v_g}\cdot\nabla\left(\frac{1}{f_0}\nabla^2\Phi+f\right)\approx F_v(p)\sin kx\cos ly \qquad -\frac{f_0^2}{\sigma}\overrightarrow{v_g}\cdot\nabla\left(-\frac{\partial\Phi}{\partial p}\right)\approx F_T(p)\sin kx\cos ly$$

we can substitute in the geopotential equation, obtaining (∂σ/∂p neglected for simplicity):

$$\frac{d^2X}{dp^2}-M^2X=\frac{\sigma}{f_0^2}\left(F_v-\frac{dF_T}{dp}\right), \qquad M^2=\frac{\sigma}{f_0^2}\left(k^2+l^2\right)$$

which is the equation for the vertical dependence X of χ

Forcing at a given altitude will generate a response whose vertical scale (measured in pressure units) is M^{-1}.

For example, upper-level vorticity advection associated with **disturbances of large horizontal scale** (small k and l) will produce **geopotential tendencies that extend down to the surface** with little loss of amplitude, while for **disturbances of small horizontal scale the response is confined close to the levels of forcing**.

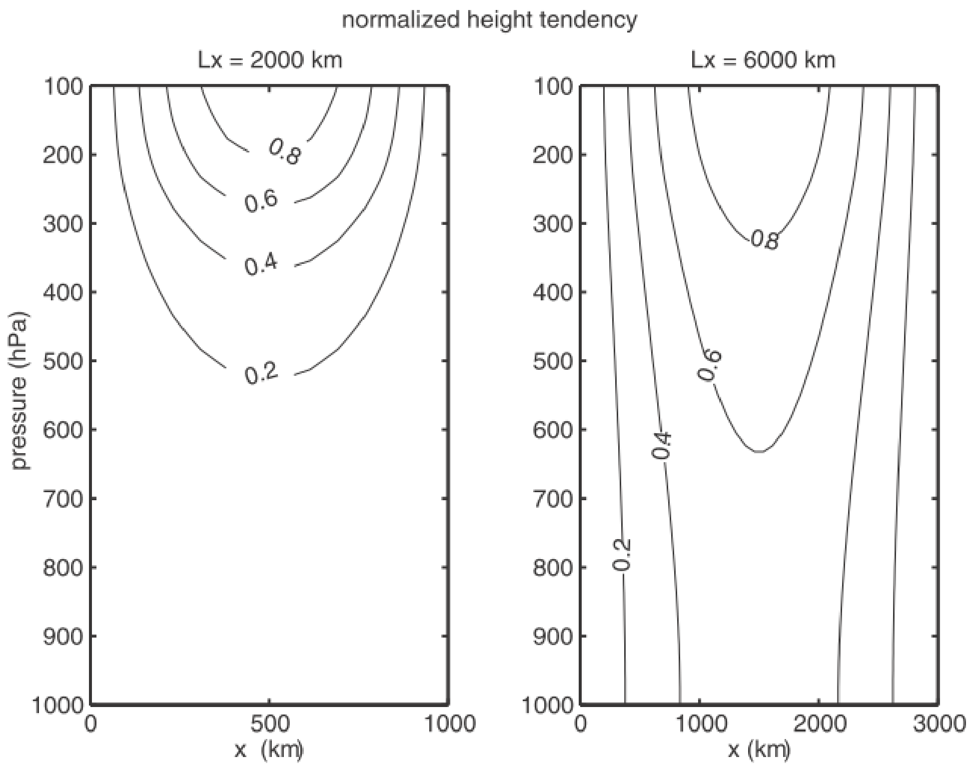


Fig. 6.11 Vertical spread of geopotential height tendencies forced by advection of a potential vorticity anomaly that is confined above the 250-hPa level. The displayed height tendencies are normalized by their value at the 100-hPa level. Solutions for zonal wavelength 2000 km (left) and 6000 km (right). Here, $l = 0$, $k = 2\pi/L_x$, $f_0 = 10^{-4}\mathrm{s}^{-1}$, and $\sigma = 2 \times 10^{-6}\mathrm{m}^2\,\mathrm{Pa}^{-2}\,\mathrm{s}^{-2}$.

In mathematical terms, operator of equation:

$$\frac{d^2X}{dp^2} - M^2X = \frac{\sigma}{f_0^2}\left(F_v - \frac{dF_T}{dp}\right)$$

spreads response in vertical so that forcing at one altitude influence other altitudes

Large scale disturbances:
→ large wavelength L_x
→ small M
→ big propagation in vertical

The role of thermal advection in changing upper-level geopotential heigths can be simply illustrated by considering the special case $\beta=0$ and very large horizontal scales so that M→0 and also $F_V \approx 0$, in order to approximate geopotential tendency equation in:

$$\frac{d^2 X}{dp^2} \approx -\frac{\sigma}{f_0^2}\frac{dF_T}{dp}$$

Integrating twice with respect to pressure: $X(p) - X(p_0) \approx -\frac{\sigma}{f_0^2}\int_{p_0}^{p} F_T\, dp$

Using definitions of X and F_T :

$$\frac{\partial}{\partial t}\left[\Phi(p) - \Phi(p_0)\right] = \frac{\partial}{\partial t}\delta\Phi \approx -R\int_{p}^{p_0} \overrightarrow{v_g}\cdot\nabla T\, d\ln p$$

Thickness tendency in the column between pressure levels p and p_0 is proportional to the vertically integrated temperature advection

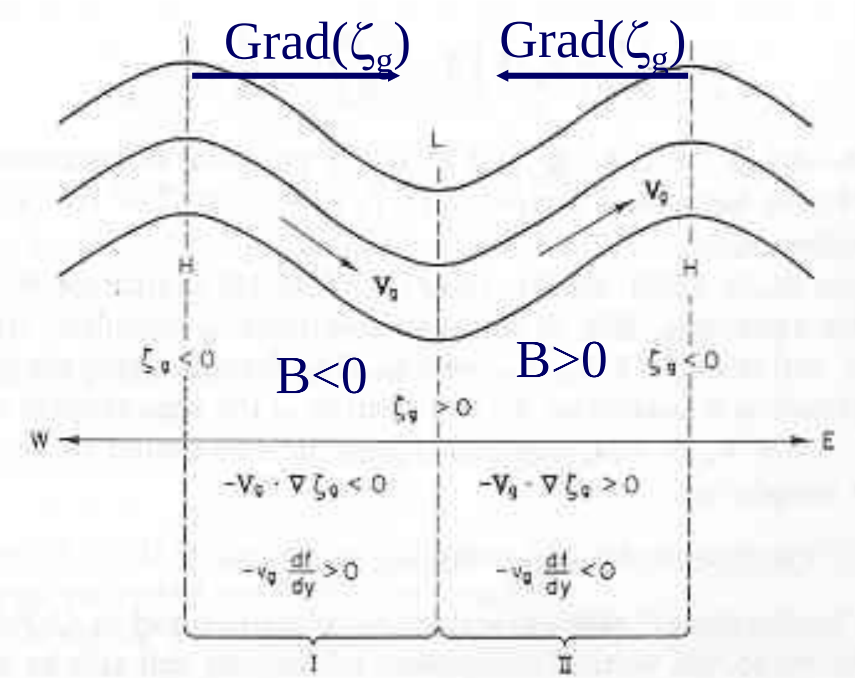
Warm advection: $\vec{v}_g \cdot \nabla T < 0$ → - $\vec{v}_g \cdot \nabla T > 0$ → $\Delta\chi > 0$ → χ increases

See slide 10

Cold advection: $\vec{v}_g \cdot \nabla T > 0$ → - $\vec{v}_g \cdot \nabla T < 0$ → $\Delta\chi < 0$ → χ decreases

$$\left[\nabla^2+\frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi = -f_0\vec{v_g}\cdot\nabla\left(\frac{1}{f_0}\nabla^2\Phi+f\right)-\frac{\partial}{\partial p}\left[-\frac{f_0^2}{\sigma}\vec{v_g}\cdot\nabla\left(-\frac{\partial\Phi}{\partial p}\right)\right]$$

A B C



Term B, i.e. vorticity advection, is generally **the main forcing term in the upper troposphere**. For <u>short waves</u>, term B is **negative in region I** (upstream of 500 hPa trough). Since sign of geopotential tendency is opposite to that of the forcing in this case, χ will be **positive** and **a ridge** will tend to develop. This ridging is of course necessary for the development of a negative geostrophic vorticity.

Similar arguments but with sign reversed apply to **region II** (downstream from 500 hPa trough), where falling geopotential heigths are associated with **positive relative vorticity advection**. Note that vorticity advection term is zero along both trough and ridge axes since laplacian of vorticity and v_g are zero. Thus, vorticity advection cannot change strength of this type of disturbance at the levels where advection is occurring, but only acts to propagate disturbances horizontally and spread vertically.

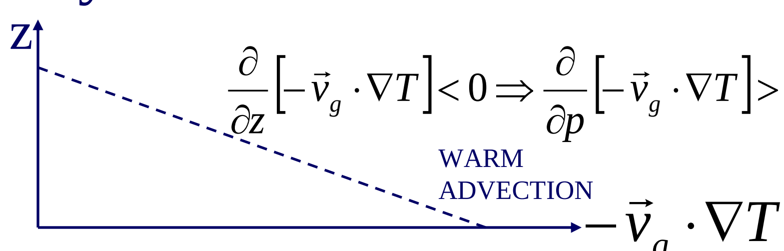
$$\underbrace{\left[\nabla^2+\frac{\partial}{\partial p}\left(\frac{f_0^{\,2}}{\sigma}\frac{\partial}{\partial p}\right)\right]\chi}_{A} = \underbrace{-f_0\overrightarrow{v_g}\cdot\nabla\left(\frac{1}{f_0}\nabla^2\Phi+f\right)}_{B}\underbrace{-\frac{\partial}{\partial p}\left[-\frac{f_0^{\,2}}{\sigma}\overrightarrow{v_g}\cdot\nabla\left(-\frac{\partial\Phi}{\partial p}\right)\right]}_{C}$$

Major mechanism for amplification or decay of midlatitude synoptic systems is contained in **<u>term C</u>**, which involves the rate of change with pressure of the horizontal thickness advection.

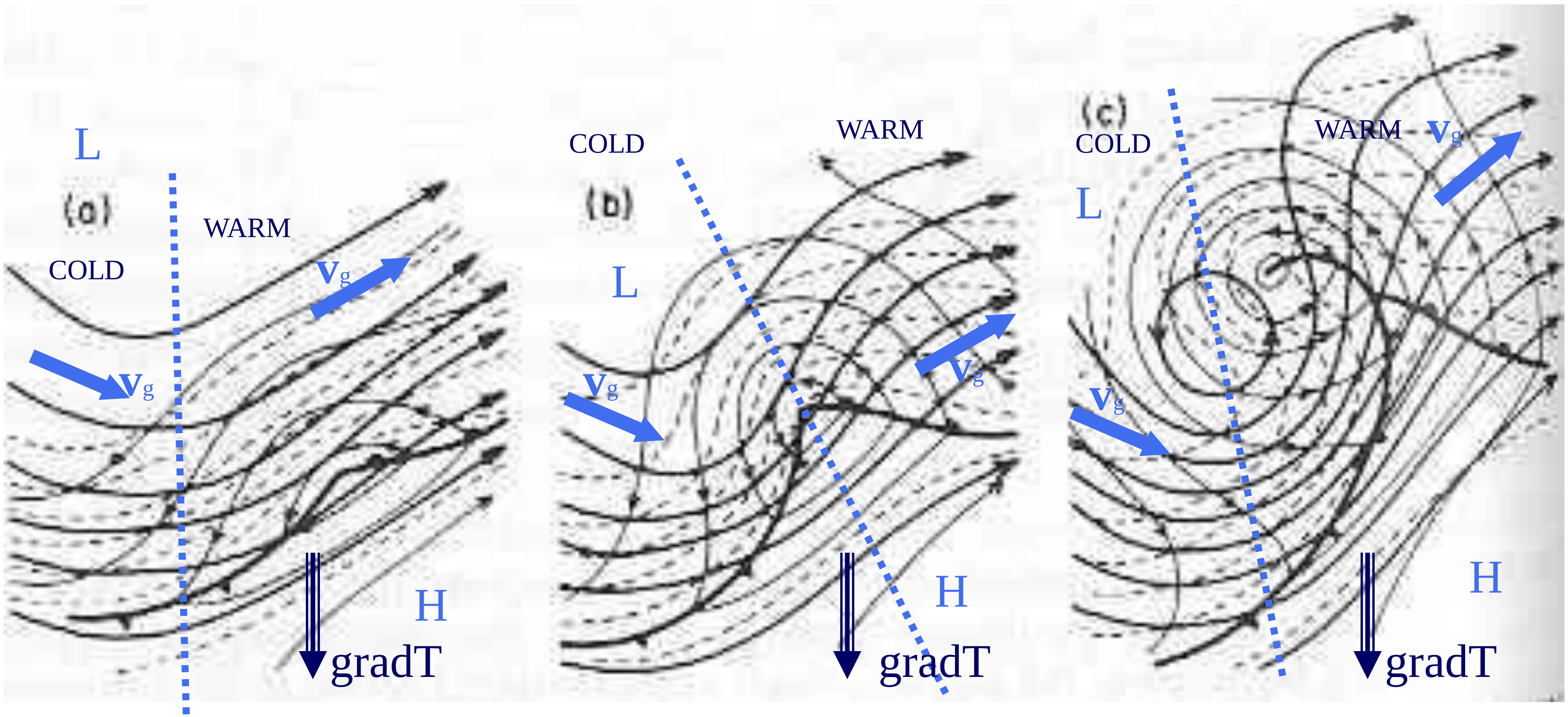
Since -∂Φ/∂p is proportional to temperature, **thickness advection is proportional to temperature advection**. Thus, term C is proportional to minus the rate of change of temperature advection with respect to the pressure (i.e. rate of change versus height).

In the idealized developing wave, **below 500 hPa ridge there is strong warm air advection associated with warm front**, while **below 500 hPa trough there is strong cold air advection associated with cold front**. Above 500 hPa level temperature gradient is usually weaker and isotherms are nearly parallel to isolines.

$$\frac{\partial}{\partial z}\left[-\vec{v}_g\cdot\nabla T\right]<0\Rightarrow\frac{\partial}{\partial p}\left[-\vec{v}_g\cdot\nabla T\right]>0$$



Then, in contrast to term B, term C tends to be concentrated in lower troposphere. Remember that geopotential tendency response is not limited to the level of forcing, but it spreads in vertical, deepening upper-level troughs and ridges



$$-\vec{v}_g \cdot \nabla T \begin{matrix} < 0 & \text{zone I (cold)} \\ > 0 & \text{zone II (warm)} \end{matrix}$$

In the region of warm advection, $-\vec{v}_g \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right) > 0$ since $\mathbf{v}_g$ has a component down to temperature gradient.

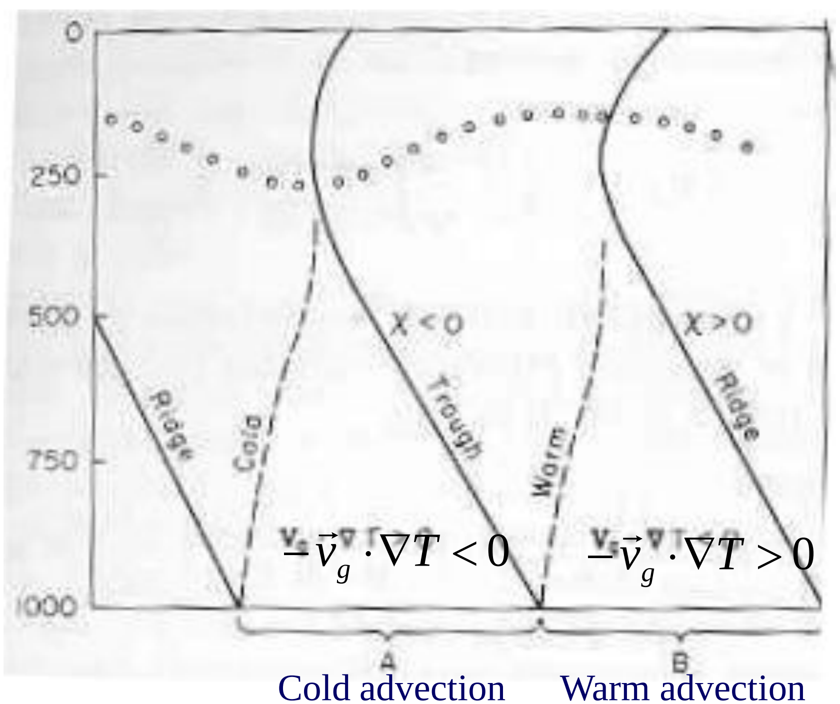
As warm advection decreases with height (increases with pressure), the derivative

$$\frac{\partial}{\partial p}\left[-\vec{v}_g \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right)\right] > 0$$

Conversely, beneath 500-hPa trough, where there is cold advection decreasing with height, opposite sign obtain.

Therefore, along 500 hPa trough and ridge axes where the vorticity advection is zero, tendency equation states that for a developing wave:

Relation between temperature advection and upper-level height tendencies



$$\chi \approx \frac{\partial}{\partial p}\left[-\vec{v}_g \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right)\right] \begin{cases} > 0 & \text{at the ridge} \\ < 0 & \text{at the trough} \end{cases}$$

Effect of cold advection below the 500 hPa trough is to deepen trough in the upper troposphere, and **effect of warm advection below 500 hPa ridge is to build the ridge** in the upper troposphere.

Hence, **differential temperature or thickness advections**, even if limited to lower troposphere, **intensifies upper-level troughs and ridges in a developing system**

- Qualitatively, effects of differential temperature advection are easily understood since **the advection of cold air into the air column below the 500 hPa trough reduces the thickness of that column and hence lowers the height of the 500 hPa surface** <u>unless there is a compensating rise in the surface pressure</u>
- Obviously, **warm advection into the air column below the ridge has the opposite effect**
- Remembering that in this derivation diabatic heating has been assumed zero, we can summarize above results by stating that **the horizontal temperature advection must be non-zero in order that a mid-latitude synoptic system intensify through baroclinic processes**
- Temperature advection patter described above indirectly implies conversion of potential energy to kinetic energy