Journal of Physics Communications

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To cite this article: M Onorato and G Dematteis 2020 *J. Phys. Commun.* **4** 095016

IOP Publishing J. Phys. Commun. 4 (2020) 095016 https://doi.org/10.1088/2399-6528/abb4b7

Journal of Physics Communications

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OPEN ACCESS

RECEIVED
12 June 2020

REVISED
27 August 2020

ACCEPTED FOR PUBLICATION
2 September 2020

PUBLISHED
16 September 2020

PAPER

A straightforward derivation of the four-wave kinetic equation in action-angle variables

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Keywords: Wave Turbulence, Wave Kinetic equation, nonlinear dispersive waves

Abstract

Starting from action-angle variables and using a standard asymptotic expansion, we present an original and coincise derivation of the Wave Kinetic equation for a resonant process of the type $2 \leftrightarrow 2$. Despite not being more rigorous than others, our procedure has the merit of being straightforward; it allows for a direct control of the random phases and random action of the initial wave field. We show that the Wave Kinetic equation can be derived assuming only initial random phases. The random action approximation has to be taken only after the weak nonlinearity and large box limits are taken. The reason is that the oscillating terms in the evolution equation for the action contain, as an argument, the action-dependent nonlinear corrections which is dropped, using the large box limit. We also show that a discrete version of the Wave Kinetic Equation can be obtained for the Nonlinear Schrödinger equation; this is because the nonlinear frequency correction terms give a zero contribution and the large box limit is not needed. In our calculation we do not make an explicitly use of the Wick selection rule.

The statistical description of a system of interacting waves is a topic of major relevance for all the fields in physics characterized by waves propagating nonlinearly. The Wave Kinetic (WK) equation [1] offers an important tool for describing those systems in and out of equilibrium; it finds application in many different fields such as as gravity, capillary and internal waves, plasma waves, Bose–Einstein condensation, elastic plate waves, etc. At the moment there is no derivation of the WK equation that can be considered as rigorous in a mathematical sense. However, physicists have attempted different roads: two are the main procedures. The first one is the direct derivation of the WK equation by performing statistical averaging over the equations of motion [2–8]; the other is through the calculation of the first moment of the equation for the probability density function for the amplitudes and phases [9–12]. Each derivation has its own strengths; at the same time, none of them seems to be adequately rigorous. In particular, while the first kind of derivation assumes the *propagation of chaos* to justify the validity of the equation at positive times, the second tries to go beyond and to prove that independent uniform phases and independent amplitudes (RPA assumption) of the initial field are sufficient to preserve the RPA hypothesis at later times. Though, none of the mentioned derivations has made an attempt (possible, in principle) to control rigorously the remainder terms of the small-ϵ perturbation expansion. The hope is that these higher order terms give a negligible contribution in the small-ϵ limit, in analogy to what has been proved in the low-density limit for gases (see for example [13]).

Our derivation does not pretend to be more rigorous than the existing ones; however, according to us, it has the merit of being straightforward. It is based on a direct expansion of the variables angle and action in powers of the small parameter in front of the interaction Hamiltonian. Because we use angle-action variables, we are able control in a clear way the two different procedures of averaging, i.e., over initial angles and over initial actions. It turns out that the phase average has to be taken before the action one; a consistent action average procedure can only be taken after the large box limit: this is because the nonlinear, amplitude dependent, frequency shift is contained as an argument of oscillating functions that appear in the equation for the amplitudes. Only in the large box limit, such term can be neglected and the action average can be safely taken. With respect to other derivations where an auxiliary intermediate time scale is introduced, see [1], in our derivation such time scale arises naturally from the expansion. It is sufficient to average over angles the evolution equation for the actions to show that the time scale of the evolution of the action variable scales like $1/\epsilon^2$, where ϵ is the perturbation parameter; action averaging is not necessary to derive the Wave Kinetic equation.

Despite the quite large literature on the subject, we feel that a didactic derivation of the four-wave kinetic equation is still lacking and we hope that newcomers in the field may benefit from our approach.

1. The hamiltonian model

We consider a Hamiltonian system with quartic nonlinearity in the Hamiltonian which allows, in the thermodynamic limit, for resonances of the type $2 \leftrightarrow 2$.

The starting physical space is defined as $\Lambda = [0, L]^d \in \mathbb{R}^d$. Its dual is the infinite discrete Fourier space $\Lambda^* = \frac{2\pi}{L}\mathbb{Z}^d$. Some shorthand notation:

$$\sum_{1234} := \sum_{k_1,k_2,k_3,k_4} \quad \text{with} \quad k_i \in \Lambda^*, \qquad \delta_{34}^{12} := \delta_{k_1+k_2,k_3+k_4} (\text{Krönecker delta}),$$
$$y_1 := y(k_1), \qquad \Delta y_{12}^{34} := y_1 + y_2 - y_3 - y_4.$$

Summations go from $-\infty$ to $+\infty$.

In normal variable a_k the Hamiltonian takes the following form:

$$\mathcal{H} = \sum_{k_1} \omega_1 |a_1|^2 + \frac{\epsilon}{2} \sum_{1234} T_{1234} a_1^* a_2^* a_3 a_4 \delta_{12}^{34}, \tag{1}$$

where $\omega_k = \omega_{-k} \geqslant 0$ is the dispersion relation and $\epsilon \ll 1$ is the small nonlinearity parameter. The equation of motion associated with the Hamiltonian is the following:

$$i\frac{da_1}{dt} = \omega_1 a_1 + \epsilon \sum_{234} T_{1234} a_2^* a_3 a_4 \delta_{12}^{34}, \tag{2}$$

that is known in many fields, for example in surface gravity waves, as the Zakharov equation, [14]. For $T_{1234} = const$ and $\omega(k) = k^2$, the equation reduces to the Nonlinear Schrödinger equation.

Performing the following transformation

$$a_k = \sqrt{I_k} \exp(-i\theta_k), \tag{3}$$

the Hamiltonian can be written in canonical action-angle variables, $\{I_k, \theta_k\}$, as

$$\mathcal{H} = \sum_1 \omega_1 I_1 + \frac{\epsilon}{2} \sum_{1234} T_{1234} \sqrt{I_1 I_2 I_3 I_4} \cos(\Delta\theta_{12}^{34}) \delta_{12}^{34}. \tag{4}$$

Hamilton's equations take the form:

$$\begin{cases} \frac{dI_1}{dt} = -\frac{\partial \mathcal{H}}{\partial \theta_1} = 2\epsilon \sum_{234} T_{1234} \sqrt{I_1 I_2 I_3 I_4} \sin(\Delta\theta_{12}^{34}) \delta_{12}^{34} \\ \frac{d\theta_1}{dt} = \frac{\partial \mathcal{H}}{\partial I_1} = \omega_1 + \epsilon \sum_{234} T_{1234} \sqrt{\frac{I_2 I_3 I_4}{I_1}} \cos(\Delta\theta_{12}^{34}) \delta_{12}^{34} \end{cases} \tag{5}$$

with initial data:

$$I_1(t=0) = \bar{I}_1, \qquad \theta_1(t=0) = \bar{\theta}_1 \tag{6}$$

To avoid secular growth in the perturbation expansion described below, the renormalized dispersion relation is introduced, see [1], [15]:

$$\Omega_1 = \omega_1 + \epsilon \left(2 \sum_2 T_{1212} I_2 - T_{1111} I_1 \right), \tag{7}$$

and the diagonal terms are extracted from the sums. As explained in [1], the introduction of the renormalized dispersion relation is fundamental for the self-consistency of the derivation of the Wave Kinetic Equation (see exercise 6.11 in [1]). If not included in the renormalized dispersion relation, the nonlinear correction to the frequency would lead to an unphysical secular growth in the expansion. In standard perturbation theory such technique is known as the Poincaré-Lindstedt method: in addition to expressing the solution itself as an asymptotic series, one expands the frequency as well and scales the time with it.

Taking into acount the expression in (7), the Hamilton's equations take the form:

$$\begin{cases} \frac{dI_1}{dt} = -\frac{\partial \mathcal{H}}{\partial \theta_1} = 2\epsilon \sum_{234}{}' T_{1234}\sqrt{I_1 I_2 I_3 I_4}\,\sin(\Delta\theta_{12}^{34})\delta_{12}^{34} \\ \frac{d\theta_1}{dt} = \frac{\partial \mathcal{H}}{\partial I_1} = \Omega_1 + \epsilon \sum_{234}{}' T_{1234}\sqrt{\frac{I_2 I_3 I_4}{I_1}}\cos(\Delta\theta_{12}^{34})\delta_{12}^{34} \end{cases} \tag{8}$$

where the sum $\sum_{234}'$ excludes all cases for which either $k_3 = k_1$ and $k_2 = k_4$, or $k_4 = k_1$ and $k_2 = k_3$, or $k_2 = k_3 = k_4 = k_1$ which are known as trivial resonances.

2. The ϵ-expansion

We perform the small-ϵ power expansion

$$\begin{aligned} I_k(t) &= I_k^{(0)}(t) + \epsilon I_k^{(1)}(t) + \epsilon^2 I_k^{(2)}(t) + \mathcal{O}(\epsilon^3) \\ \theta_k(t) &= \theta_k^{(0)}(t) + \epsilon\theta_k^{(1)}(t) + \epsilon^2\theta_k^{(2)}(t) + \mathcal{O}(\epsilon^3) \end{aligned} \tag{9}$$

and plug into (8) to obtain, order by order,

- ϵ^0:
  Linear evolution where only the fast angle oscillations are at play,

$$\begin{cases} \frac{dI_k^{(0)}}{dt} = 0 \\ \frac{d\theta_k^{(0)}}{dt} = \bar{\Omega}_k \end{cases} \Rightarrow \quad \begin{cases} I_k^{(0)}(t) = \bar{I}_k = \text{const} \\ \theta_k^{(0)} = \bar{\theta}_k + \bar{\Omega}_k t \mod 2\pi \end{cases} \tag{10}$$

  Here $\bar{\Omega}_k = \omega_k + \epsilon(2\sum_{k_2} T_{kk_2kk_2}\bar{I}_{k_2} - T_{kkkk}\bar{I}_k)$, with $\bar{I}_k = I_k(t=0) = I_k^{(0)}(t=0)$. While the angles evolve on the linear time scale, the variations for the actions require a higher order dynamics in ϵ. Note also that the linear time scale $1/\bar{\Omega}_k$ is k dependent; this implies that for example for dispersion relations for which $\bar{\Omega}_k \to 0$ for $k \to 0$, then the linear time scale may become extremely large.

- ϵ^1:

$$\begin{cases} \frac{dI_1^{(1)}}{dt} = 2\sum_{234}' T_{1234}\sqrt{\bar{I}_1\bar{I}_2\bar{I}_3\bar{I}_4}\,\sin(\Delta\theta^{(0)}{}_{12}^{34})\delta_{12}^{34} \\ \frac{d\theta_1^{(1)}}{dt} = \sum_{234}' T_{1234}\sqrt{\frac{\bar{I}_2\bar{I}_3\bar{I}_4}{\bar{I}_1}}\cos(\Delta\theta^{(0)}{}_{12}^{34})\delta_{12}^{34} \end{cases} \tag{11}$$

  Integrating in time from 0 to t, yields

$$\begin{cases} I_1^{(1)} = 2\sum_{234}' T_{1234}\sqrt{\bar{I}_1\bar{I}_2\bar{I}_3\bar{I}_4}\,\frac{\cos(\Delta\bar{\theta}_{12}^{34}) - \cos(\Delta\bar{\theta}_{12}^{34} + \Delta\bar{\Omega}_{12}^{34}t)}{\Delta\bar{\Omega}_{12}^{34}}\delta_{12}^{34} \\ \theta_1^{(1)} = \sum_{234}' T_{1234}\sqrt{\frac{\bar{I}_2\bar{I}_3\bar{I}_4}{\bar{I}_1}}\,\frac{\sin(\Delta\bar{\theta}_{12}^{34} + \Delta\bar{\Omega}_{12}^{34}t) - \sin(\Delta\bar{\theta}_{12}^{34})}{\Delta\bar{\Omega}_{12}^{34}}\delta_{12}^{34} \end{cases} \tag{12}$$

- ϵ^2:

$$\begin{aligned} \frac{dI_1^{(2)}}{dt} = 2\sum_{234}' T_{1234}\sqrt{\bar{I}_1\bar{I}_2\bar{I}_3\bar{I}_4}\Bigg[\frac{1}{2}\left(\frac{I_1^{(1)}}{\bar{I}_1} + \frac{I_2^{(1)}}{\bar{I}_2} + \frac{I_3^{(1)}}{\bar{I}_3} + \frac{I_4^{(1)}}{\bar{I}_4}\right)\sin(\Delta\bar{\theta}_{12}^{34} + \Delta\bar{\Omega}_{12}^{34}t) \\ + \Delta\theta^{(1)}{}_{12}^{34}\cos(\Delta\bar{\theta}_{12}^{34} + \Delta\bar{\Omega}_{12}^{34}t)\Bigg]\delta_{12}^{34} \end{aligned} \tag{13}$$

  which substituting the expressions in (12) leads, after some algebra and the use of trigonometric identities, to the compact form

$$\begin{aligned} \frac{dI_1^{(2)}}{dt} &= 2\sum_{234}'\sum_{567}'\sum_{m=1}^{4} T_{1234}T_{m567}\frac{1}{\sqrt{\bar{I}_m}}\sqrt{\bar{I}_1\bar{I}_2\bar{I}_3\bar{I}_4\bar{I}_5\bar{I}_6\bar{I}_7} \\ &\times\left[\frac{\sin[\sigma_m\Delta\bar{\theta}_{m5}^{67} - \Delta\bar{\theta}_{12}^{34} + (\sigma_m\Delta\bar{\Omega}_{m5}^{67} - \Delta\bar{\Omega}_{12}^{34})t] + \sin[\Delta\bar{\theta}_{12}^{34} - \sigma_m\Delta\bar{\theta}_{m5}^{67} + \Delta\bar{\Omega}_{12}^{34}t]}{\Delta\bar{\Omega}_{m5}^{67}}\right]\delta_{12}^{34}\delta_{m5}^{67}, \end{aligned} \tag{14}$$

  where $\sigma = (+1, +1, -1, -1)$. The evolution equation for $\theta_k^{(2)}$ is not needed for the derivation of the Wave Kinetic equation.

For the evolution of the action variable we have thus obtained:

$$\frac{dI_k}{dt} = \epsilon \frac{dI_k^{(1)}}{dt} + \epsilon^2 \frac{dI_k^{(2)}}{dt} + \mathcal{O}(\epsilon^3), \quad (15)$$

where the terms on the right hand side are given respectively by (11) and (14).

3. Averaging over initial angles: the discrete wave kinetic equation

We define the procedure of averaging over initial phases, $\bar{\theta}_k$, of the observable $g(\bar{\theta}_1, \bar{\theta}_2, \ldots, \bar{\theta}_N)$ as:

$$\langle g(\bar{\theta}_1, \bar{\theta}_2, \ldots, \bar{\theta}_N)\rangle_{\bar{\theta}} = \int_0^{2\pi} P(\bar{\theta}_1, \bar{\theta}_2, \ldots, \bar{\theta}_N) g(\bar{\theta}_1, \bar{\theta}_2, \ldots, \bar{\theta}_N) d\bar{\theta}_1 d\bar{\theta}_2 \ldots d\bar{\theta}_N \quad (16)$$

where $P(\bar{\theta}_1, \bar{\theta}_2, \ldots, \bar{\theta}_N)$ is the joint probability density function. Assuming that phases are statistically independent, then $P(\bar{\theta}_1, \bar{\theta}_2, \ldots, \bar{\theta}_N) = P(\bar{\theta}_1)P(\bar{\theta}_2)\ldots P(\bar{\theta}_N)$; moreover, phases are uniformly distributed so that the average is computed as follows:

$$\langle g(\bar{\theta}_1, \bar{\theta}_2, \ldots, \bar{\theta}_N)\rangle_{\bar{\theta}} = \frac{1}{(2\pi)^N} \int_0^{2\pi} g(\bar{\theta}_1, \bar{\theta}_2, \ldots, \bar{\theta}_N) d\bar{\theta}_1 d\bar{\theta}_2 \ldots d\bar{\theta}_N \quad (17)$$

We are interested in the following:

$$\left\langle \frac{dI_k}{dt} \right\rangle \bar{\theta} = \epsilon \left\langle \frac{dI_k^{(1)}}{dt} \right\rangle \bar{\theta} + \epsilon^2 \left\langle \frac{dI_k^{(2)}}{dt} \right\rangle \bar{\theta} + \mathcal{O}(\epsilon^3), \quad (18)$$

Two time scales appear in equation (18); however, as it will be clear soon, the procedure of averaging over the initial phases makes the shortest time scale term vanish:

- ϵ

$$\frac{d\langle I_1^{(1)}\rangle_{\bar{\theta}}}{dt} = 2\left\langle \sum_{234}' T_{1234}\sqrt{\bar{I}_1\bar{I}_2\bar{I}_3\bar{I}_4}\,\sin(\Delta\theta_{12}^{34})\delta_{12}^{34} \right\rangle_{\bar{\theta}} \quad (19)$$

  Using the complex exponential notation, we get:

$$\frac{d\langle I_1^{(1)}\rangle_{\bar{\theta}}}{dt} = 2\Im\left[\sum_{234}' T_{1234}\sqrt{\bar{I}_1\bar{I}_2\bar{I}_3\bar{I}_4}\,\langle \exp[i\Delta\bar{\theta}\,{}_{12}^{34}\,]\rangle_{\bar{\theta}}\ \exp[i\Delta\bar{\Omega}\,{}_{12}^{34}t\,]\delta_{12}^{34}\right], \quad (20)$$

  where $\Im$ stands for the imaginary part. The r.h.s. is 0 because $\langle\, \exp[i\Delta\bar{\theta}\,{}_{12}^{34}\,]\rangle_{\bar{\theta}} = 0$:

$$\frac{d\langle I_1^{(1)}\rangle_{\bar{\theta}}}{dt} = 0 \quad (21)$$

  This implies that the action evolution depends, after phase averaging, on higher order contributions.

- ϵ^2

  Using the complex exponential notation, equation (14) can be rewritten as:

$$\frac{d\langle I_1^{(2)}\rangle_{\bar{\theta}}}{dt} = 2\Im\Bigg[\Sigma_{234}'\Sigma_{567}' \sum_{m=1}^{4} T_{1234} T_{m567} \frac{1}{\sqrt{\bar{I}_m}} \sqrt{\bar{I}_1\bar{I}_2\bar{I}_3\bar{I}_4\bar{I}_5\bar{I}_6\bar{I}_7} \times \frac{(\exp[i(\sigma_m\Delta\bar{\Omega}_{m5}^{67}t] - 1)\exp[-i\Delta\bar{\Omega}_{12}^{34})t)]}{\Delta\bar{\Omega}_{m5}^{67}} \langle \exp[i(\sigma_m\Delta\bar{\theta}_{m5}^{67} - \Delta\bar{\theta}_{12}^{34})]\rangle_{\bar{\theta}}\delta_{12}^{34}\delta_{m5}^{67}\Bigg]. \quad (22)$$

  In order to show how the above equation simplifies after the phase averaging procedure, we consider the r.h.s for $m = 1$:

$$2\Im\Bigg[\Sigma_{234}'\Sigma_{567}'\ T_{1234}T_{1567}\sqrt{\bar{I}_2\bar{I}_3\bar{I}_4\bar{I}_5\bar{I}_6\bar{I}_7} \times \frac{(\exp[i(\Delta\bar{\Omega}_{15}^{67}t] - 1)\exp[-i\Delta\bar{\Omega}_{12}^{34})t)]}{\Delta\bar{\Omega}_{15}^{67}} \langle \exp[i\Delta\bar{\theta}_{345}^{267}]\rangle_{\bar{\theta}}\delta_{12}^{34}\delta_{15}^{67}\Bigg]. \quad (23)$$

  Due to the restrictions in the sums, the only possible values of k_j that give a contribution of the above expression different from 0, are the following: $k_5 = k_2, k_6 = k_3, k_7 = k_4$ and $k_5 = k_2, k_6 = k_4, k_7 = k_3$. For each of these two sets of wave numbers the phase average term gives a contribution of 1. The expression above reduces to

$$4\sum_{234}{}' T_{1234}^2 \bar{I}_2\bar{I}_3\bar{I}_4 \frac{\sin(\Delta\bar{\Omega}_{12}^{34}t)}{\Delta\bar{\Omega}_{12}^{34}}\delta_{12}^{34} \tag{24}$$

A similar contribution, with appropriate sign and indices, can be derived for $m = 2, 3, 4$. The final evolution equation for $\langle I_k^{(2)}\rangle_{\bar{\theta}}$ reads

$$\frac{d\langle I_1^{(2)}\rangle_{\bar{\theta}}}{dt} = 4\sum_{234} T_{1234}^2 \bar{I}_1\bar{I}_2\bar{I}_3\bar{I}_4\left(\frac{1}{\bar{I}_1} + \frac{1}{\bar{I}_2} - \frac{1}{\bar{I}_3} - \frac{1}{\bar{I}_4}\right)\frac{\sin(\Delta\bar{\Omega}_{12}^{34}t)}{\Delta\bar{\Omega}_{12}^{34}}\delta_{12}^{34} \tag{25}$$

Note that there is no need to use the reduced sum $\sum'$ symbol, because the extra terms in the standard sum give a zero contribution (exact cancellations due to the two '+' signs and two '−' signs in the term in brackets).

Inserting equation (25) into equation (18), gives

$$\frac{d\langle I_1\rangle_{\bar{\theta}}}{dt} = 4\epsilon^2\sum_{234} T_{1234}^2 \bar{I}_1\bar{I}_2\bar{I}_3\bar{I}_4\left(\frac{1}{\bar{I}_1} + \frac{1}{\bar{I}_2} - \frac{1}{\bar{I}_3} - \frac{1}{\bar{I}_4}\right)\frac{\sin(\Delta\bar{\Omega}_{12}^{34}t)}{\Delta\bar{\Omega}_{12}^{34}}\delta_{12}^{34} \tag{26}$$

If we define the nonlinear time $\tau = \epsilon^2 t$, then the equation reads

$$\frac{d\langle I_1\rangle_{\bar{\theta}}}{d\tau} = 4\sum_{234} T_{1234}^2 \bar{I}_1\bar{I}_2\bar{I}_3\bar{I}_4\left(\frac{1}{\bar{I}_1} + \frac{1}{\bar{I}_2} - \frac{1}{\bar{I}_3} - \frac{1}{\bar{I}_4}\right)\frac{\sin(\Delta\bar{\Omega}_{12}^{34}\tau/\epsilon^2)}{\Delta\bar{\Omega}_{12}^{34}}\delta_{12}^{34}. \tag{27}$$

If $T_{kk'kk'} = \text{const}$ and $\sum_k I_k$ is conserved (this property is enjoyed for instance by the Nonlinear Schrödinger equation), then the nonlinear frequency shift contribution is identically zero and, in the limit of $\epsilon \to 0$, equation (27) becomes

$$\frac{d\langle I_1\rangle_{\bar{\theta}}}{d\tau} = 4\pi\sum_{234} T_{1234}^2 \bar{I}_1\bar{I}_2\bar{I}_3\bar{I}_4\left(\frac{1}{\bar{I}_1} + \frac{1}{\bar{I}_2} - \frac{1}{\bar{I}_3} - \frac{1}{\bar{I}_4}\right)\delta(\Delta\omega_{12}^{34})\delta_{12}^{34}, \tag{28}$$

where we have used the property that

$$\lim_{\epsilon\to 0}\frac{\sin(\Delta\omega_{12}^{34}\tau/\epsilon^2)}{\Delta\omega_{12}^{34}} = \pi\delta(\Delta\omega_{12}^{34}). \tag{29}$$

Remarks

- The $\delta(\Delta\omega_{12}^{34})$ above is dimensionally a Dirac delta, coming from the limit relationship (29). This is not rigorous and in principle even not meaningful, being the argument of the δ not a continuous function. Though, one can argue that the values taken by $\Delta\omega_{12}^{34}$ can become extremely dense around $\Delta\omega_{12}^{34} = 0$, which can be thought of as the summation tending to an integral.
- The time scale for the evolution of the action is $1/\epsilon^2$.
- For the validity of the expansion, such time scale should always be much larger than the linear time scale given by $1/\omega_k$ for all values of k.
- In the r.h.s. only the initial actions are included.
- The highest order contribution to the evolution of I_k at order ϵ^2 from the angle dynamics comes from $\theta_k^{(1)}$. No contribution from $\theta_k^{(2)}$ enters.
- No assumptions on the statistics of initial actions have been made.
- Equation (28) is meaningful only if the dispersion relation allows for connected exact resonances on a regular discrete grid.

4. The thermodynamic limit: the standard wave kinetic equation

The physical space over which we have worked is defined as $\Lambda = [0, L] \in \mathbb{R}^d$. In the thermodynamic limit one is interested in looking at the limit $L \to \infty$. As this limit is taken, the spacing between Fourier modes $\Delta k = 2\pi/L$ becomes smaller and smaller in such a way that wave number space becomes dense: $k \in \mathbb{R}^d$. In this limit a resonant manifold, that could be empty in the case of regular discrete grid, may appear. Therefore, the starting point for the derivation should be equation (27), where the Dirac Delta function over frequencies has not been introduced yet. The thermodynamic limit ($\Delta k \to 0$ or $L \to \infty$) is taken using the following rules:

- We define the action density as:

$$\mathrm{I}_k = \mathrm{I}(k,\, t) := \frac{I_k}{\Delta k^d}, \tag{30}$$

where $\mathrm{I}(k,\, t)$ is a continuous function of $k \in \mathbb{R}^d$

- Sums become integrals as follows:

$$\sum_k \to \int \frac{1}{\Delta k^d} dk \tag{31}$$

- The Kronecker Delta $\delta^{(K)}$ becomes a Dirac Delta $\delta^{(D)}$

$$\delta^{(K)} \to \Delta k^d \delta^{(D)} \tag{32}$$

Introducing the above substitutions in (27), we get:

$$\frac{\partial \langle \mathrm{I}_1 \rangle_{\bar{\theta}}}{\partial \tau} = 4 \int dk_2 dk_3 dk_4 T_{1234}^2 \bar{\mathrm{I}}_1 \bar{\mathrm{I}}_2 \bar{\mathrm{I}}_3 \bar{\mathrm{I}}_4 \left(\frac{1}{\bar{\mathrm{I}}_1} + \frac{1}{\bar{\mathrm{I}}_2} - \frac{1}{\bar{\mathrm{I}}_3} - \frac{1}{\bar{\mathrm{I}}_4} \right) \frac{\sin(\Delta \bar{\Omega}_{12}^{34} \tau / \epsilon^2)}{\Delta \bar{\Omega}_{12}^{34}} \delta_{12}^{34}, \tag{33}$$

where we need to take the limit for $\Delta k \to 0$ of

$$\lim_{\Delta k \to 0} \frac{\sin\left[(\omega_k + \Delta k^d \epsilon 2 \sum_{k_2} T_{kk_2kk_2} \bar{\mathrm{I}}_2 - \epsilon\, \Delta k^d T_{kkkk} \bar{\mathrm{I}}_k) \tau / \epsilon^2 \right]}{\left(\omega_k + \Delta k^d \epsilon 2 \sum_{k_2} T_{kk_2kk_2} \bar{\mathrm{I}}_2 - \Delta k^d \epsilon\, T_{kkkk} \bar{\mathrm{I}}_k \right)} = \frac{\sin(\omega_k \tau / \epsilon^2)}{\omega_k} \tag{34}$$

The last equality is valid only if we assume that

$$\lim_{\substack{\Delta k \to 0 \\ \epsilon \to 0}} \frac{\Delta k^d}{\epsilon} = 0 \tag{35}$$

4.1. The weakly nonlinear limit

By taking the small amplitude limit $\epsilon \to 0$, satisfying equation (35), one thus gets

$$\frac{\partial \langle \mathrm{I}_1 \rangle_{\bar{\theta}}}{\partial \tau} = 4\pi \int dk_2 dk_3 dk_4 T_{1234}^2 \bar{\mathrm{I}}_1 \bar{\mathrm{I}}_2 \bar{\mathrm{I}}_3 \bar{\mathrm{I}}_4 \left(\frac{1}{\bar{\mathrm{I}}_1} + \frac{1}{\bar{\mathrm{I}}_2} - \frac{1}{\bar{\mathrm{I}}_3} - \frac{1}{\bar{\mathrm{I}}_4} \right) \delta(\Delta \omega_{12}^{34}) \delta_{12}^{34}, \tag{36}$$

4.2. Actions as stochastic variables

We now assume that I_k is a stochastic variable whose expectation value taken with respect to the distribution of the initial actions is given by

$$n(k,\, t) = \langle \mathrm{I}_k \rangle_{\bar{\theta}, \bar{\mathrm{I}}}; \tag{37}$$

the equation above defines the spectral action density $n(k,\, t)$ or more simply the action spectrum. We assume that the initial actions labeled by different wave numbers are statistically independent, so that

$$\langle \bar{\mathrm{I}}_i \bar{\mathrm{I}}_j \bar{\mathrm{I}}_k \rangle_{\bar{\mathrm{I}}} = \langle \bar{\mathrm{I}}_i \rangle_{\bar{\mathrm{I}}} \langle \bar{\mathrm{I}}_j \rangle_{\bar{\mathrm{I}}} \langle \bar{\mathrm{I}}_k \rangle_{\bar{\mathrm{I}}} = \bar{n}_i \bar{n}_j \bar{n}_k, \quad i \neq j \neq k, \tag{38}$$

where $\bar{n}_k = n(k,\, t = 0) = \langle \bar{\mathrm{I}}_k \rangle_{\bar{\mathrm{I}}}$; therefore, the equation for the spectrum becomes

$$\frac{\partial n_1}{\partial \tau} = 4\pi \int dk_2 dk_3 dk_4 T_{1234}^2 \bar{n}_1 \bar{n}_2 \bar{n}_3 \bar{n}_4 \left(\frac{1}{\bar{n}_1} + \frac{1}{\bar{n}_2} - \frac{1}{\bar{n}_3} - \frac{1}{\bar{n}_4} \right) \delta(\Delta \omega_{12}^{34}) \delta_{12}^{34}, \tag{39}$$

Again, on the right hand side of the equation only initial data for n_k are included. Thus, strictly speaking its validity is at time $t = 0$. A usual, but somehow unjustified, further step consists in substituting in the right hand side the spectral action density n_k with $\bar{n}_k = n_k(t = 0)$ to get:

$$\frac{\partial n_1}{\partial \tau} = 4\pi \int dk_2 dk_3 dk_4 T_{1234}^2 n_1 n_2 n_3 n_4 \left(\frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4} \right) \delta(\Delta \omega_{12}^{34}) \delta_{12}^{34}, \tag{40}$$

This is the celebrated Wave Kinetic equation. The substitution of $\bar{n}_k = n_k(t = 0)$ with $n_k(t)$ could be justified if random phases and amplitudes would persist during the evolution up to time scale of validity of the equation.

As a last remark, we note that the statistical assumptions needed to obtain equation (40) ultimately amount to a field with random independent uniformly distributed angles and independent actions (RPA). Regarding the relation to the assumption of (quasi) Gaussianity commonly used in earlier derivations of the Wave Kinetic Equation, as discussed in [11], we mention that all Gaussian random fields are Random Phase and Amplitude; conversely, for any sequence of Random Phase and Amplitude fields, the spatial field converges in distribution to a Gaussian field with zero mean and spectrum $n(k)$ as $L \to \infty$.

5. Conclusions

The Wave Kinetic equation is an important tool in physics; the field of research is very active both from a theoretical and experimental point of view. Establishing the validity of the Kinetic Equation in reproducing the statistical behaviour of a system of random waves is a topic of paramount importance. Therefore, a rigorous derivation of the equation would be of great benefit for the community. In this spirit, we have presented a new formal derivation of the equation based on wave-action variables. Our objective has been to make a coincise and self-consistent derivation, without loosing rigour. We have clarified that a kinetic equation for deterministic actions can be derived by using only the randomness of the initial phases. Moreover, a discrete form of the kinetic equation for deterministic actions is also derived. In general the equation contains the sinc function and only for the specific case of the NLS equation, in the limit of small ϵ, the standard Dirac δ will appear in the equation. To what extent such function is meaningful in the equation it is still to be understood. It is out of the scope of the present paper to discuss the convergence of the expansion used or the persistence of the statistics of the initial condition (see [9–12] for a discussion).

Acknowledgments

M O was supported by the 'Departments of Excellence 2018-2022' Grant awarded by the Italian Ministry of Education, University and Research (MIUR) (L.232/2016). M O was supported by Simons Collaboration on Wave Turbulence, Grant No. 617006. The authors thank S. Nazarenko for valuable discussions.

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