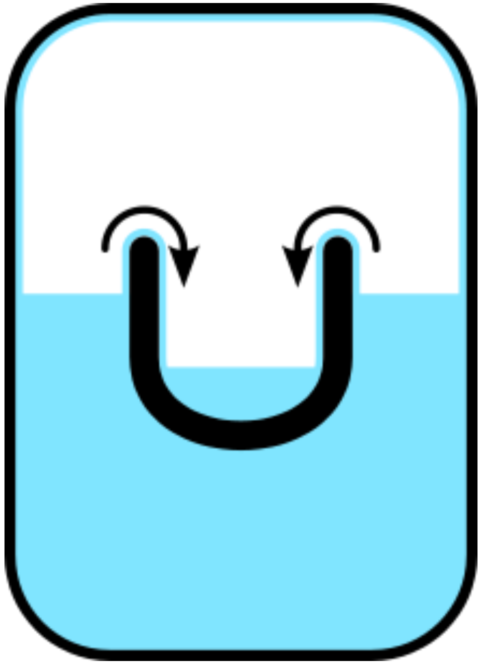
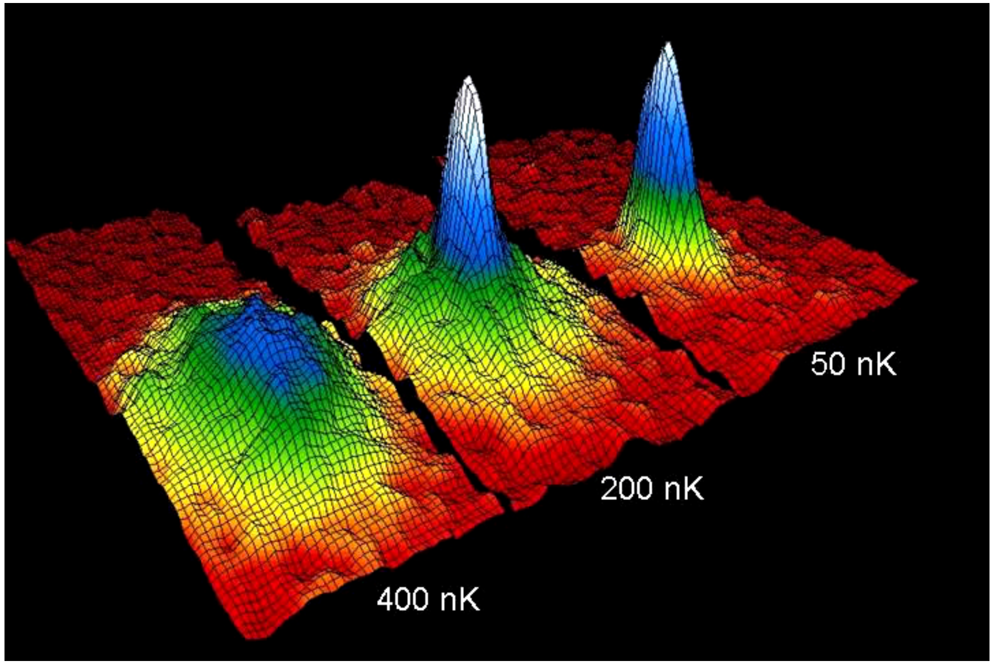
A brief history of superfluids and the Gross-Pitaevskii model



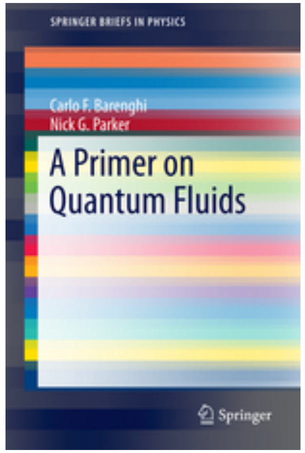


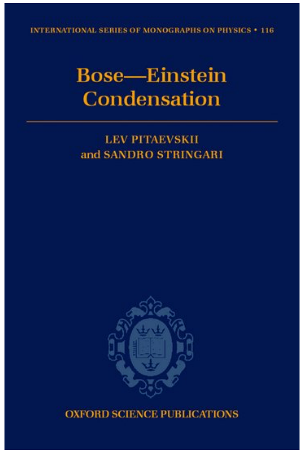
Dr Davide Proment, University of East Anglia (UEA)

Torino, 17th November 2020

Good books on the topic!

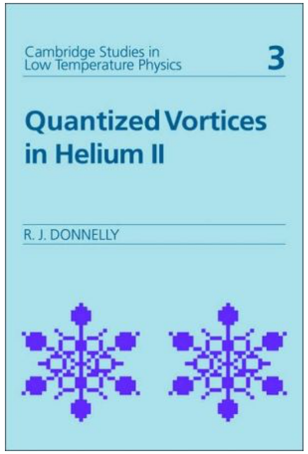
C.F. Barenghi & N.G. Parker, A Primer on Quantum Fluids, Springer, 2016.
free online, arxiv1605.09580





L.P. Pitaevskii & S. Stringari, Bose-Einstein condensation, Oxford University Press, 2003.

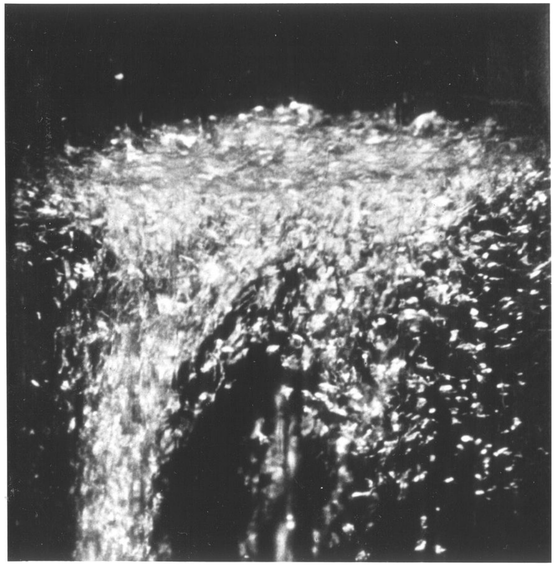
R.J. Donnelly, Quantized Vortices in Helium II, Cambridge University Press, 1991.



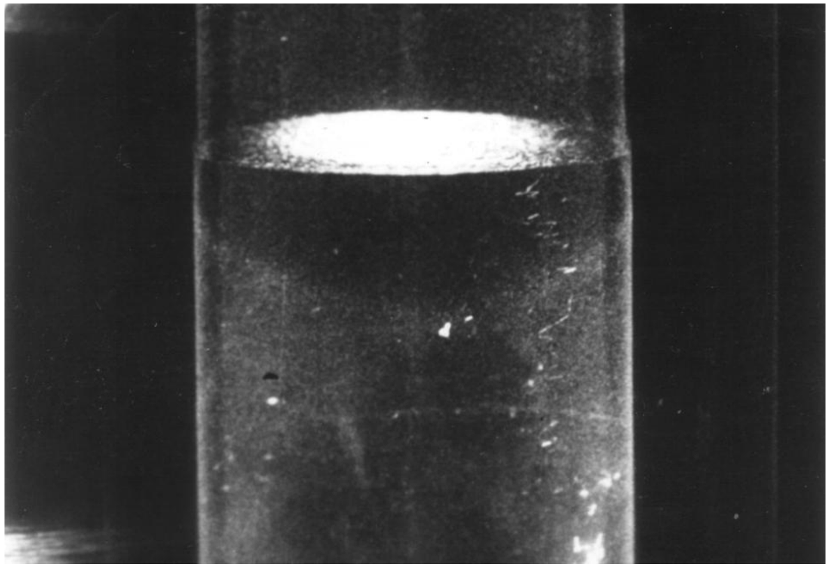
Liquefaction of helium

- At atmospheric pressure helium exists in a liquid form only at the temperature below 4K.
- Two isotope of helium is present: helium-4 (common) or helium-3 (rare).
- Helium was first liquefied in 1908 by **H.K. Onnes** at the University of Leiden.
- Because of the very weak interatomic forces, He remains a liquid at atmospheric pressure all the way down to absolute zero.

Heike Kamerlingh Onnes

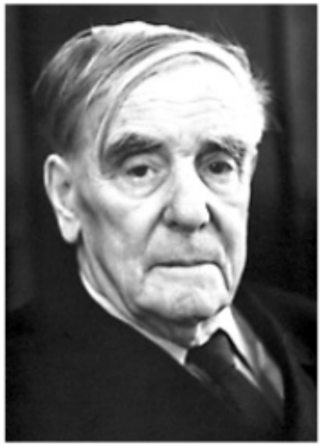


helium-4, T>2.7K at 1atm



helium-4, T<2.7K at 1atm

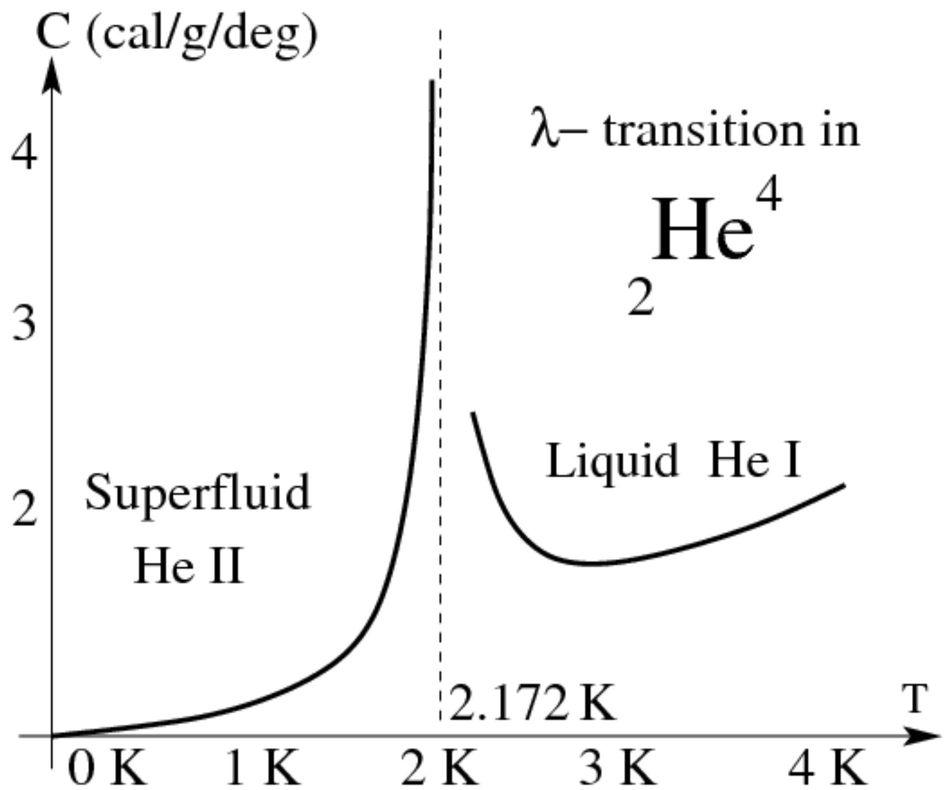
The lambda point



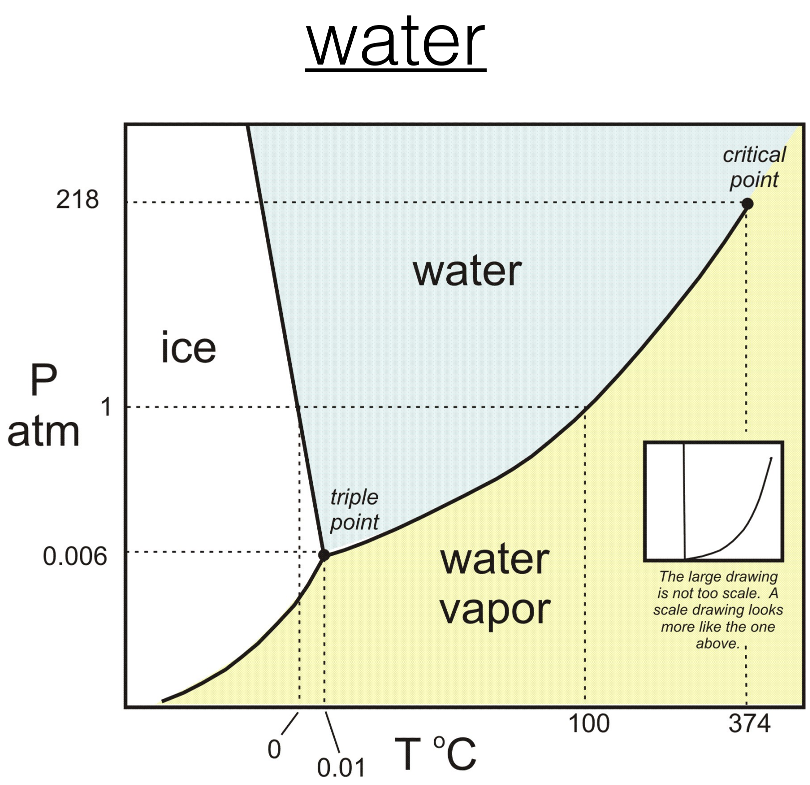
Pyotr Leonidovich Kapitsa

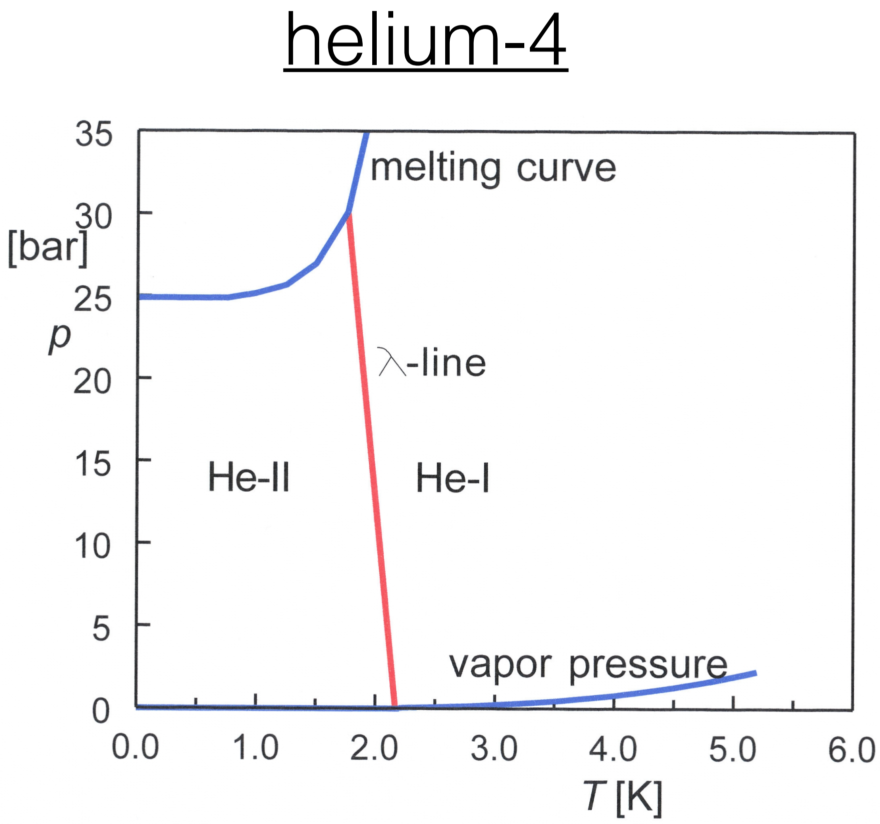
- At temperatures below their liquefaction points, both helium-4 and helium-3 undergo a transition to superfluid.
- The lambda point is the temperature at which normal fluid helium makes the transition to superfluid helium (He-II). At 1 atmosphere the temperature is approximately 2.17K for helium-4.

- The name derives from the graph that results from the plot of the specific heat capacity as a function of the temperature and its similarity with the Greek letter lambda.



Phase diagram: water vs. helium-4

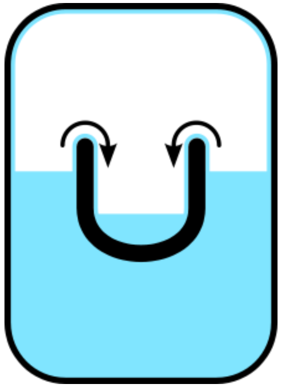




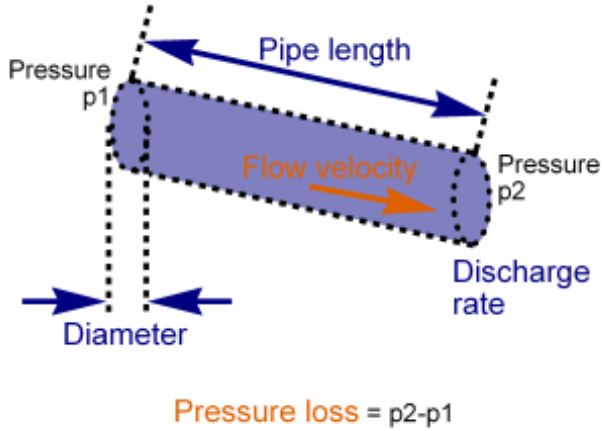
- First-order phase transitions: involve latent heat between the transition. During the transition the system releases or absorbs a fixed amount of energy per volume, and the temperature stays constant. The most common example is solid-liquid water transition. This is a consequence of the discontinuity of the free energy when varying the temperature.
- Second-order phase transitions: they are also called continuous phase transitions and they are characterised by the discontinuity of the first derivative of the free energy. Examples are ferromagnetic transition, superconducting transition, and superfluid transition.

Why He-II is a superfluid?

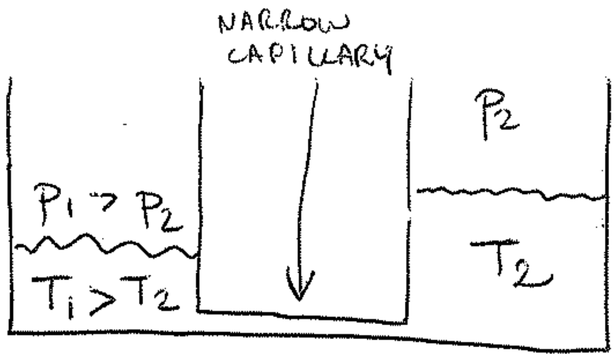
- ability to climb a tube to reach a lower level.



- flow in a pipe without experiencing a pressure drop: <u>zero viscosity</u>.



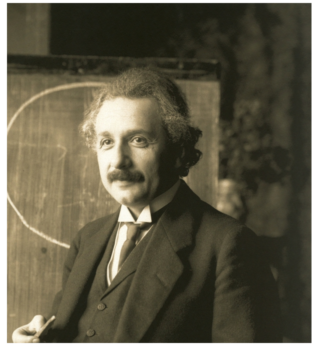
- two containers connected by a very thin pipe (where viscous fluid cannot flow) are still thermo-mechanically connected via the superfluid component.



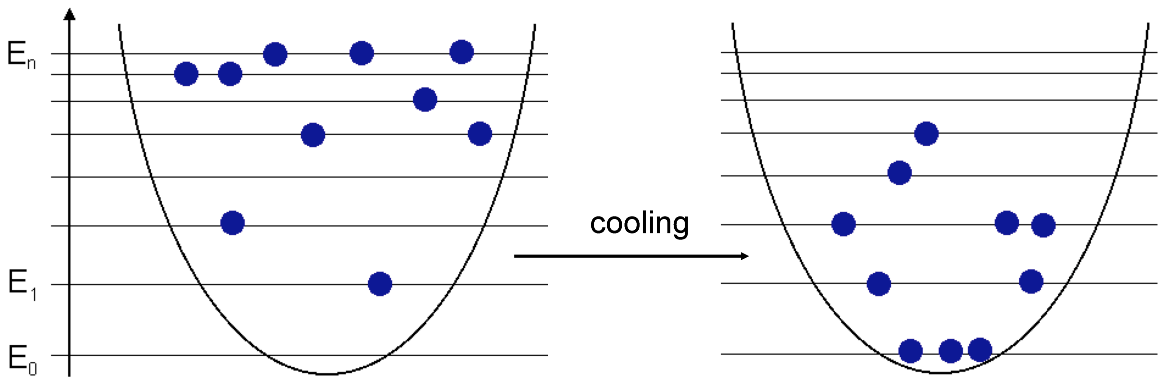
How to explain these properties?

Use the concept of Bose-Einstein condensation: at sufficiently low temperatures a system of bosons undergoes a phase transition where a macroscopic fraction of its constituents "condense" into a single wave-function called Bose-Einstein condensate.

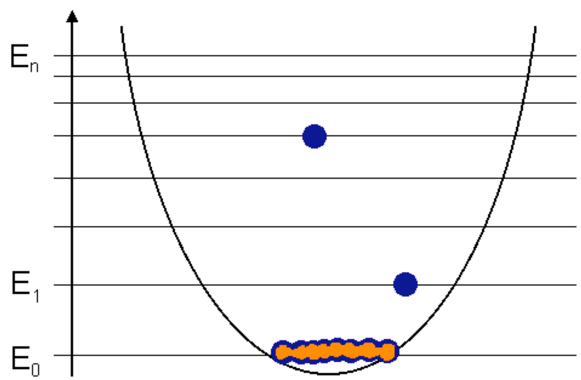
Satyendra Nath Bose



Albert Einstein

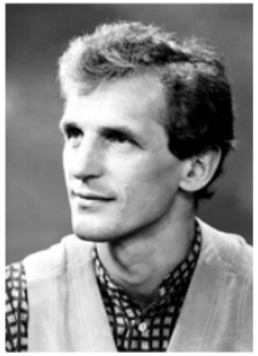


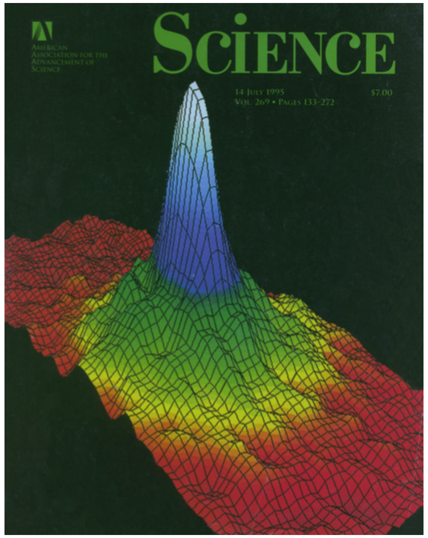
Temperatures above Tc



Temperatures below Tc

Realisation of Bose-Einstein condensates





- Observation of Bose-Einstein Condensation in a Dilute Atomic Vapor, M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, E. A. Cornell, Science, 269, 5221, 1980–201 (1995).

PHYSICAL REVIEW LETTERS

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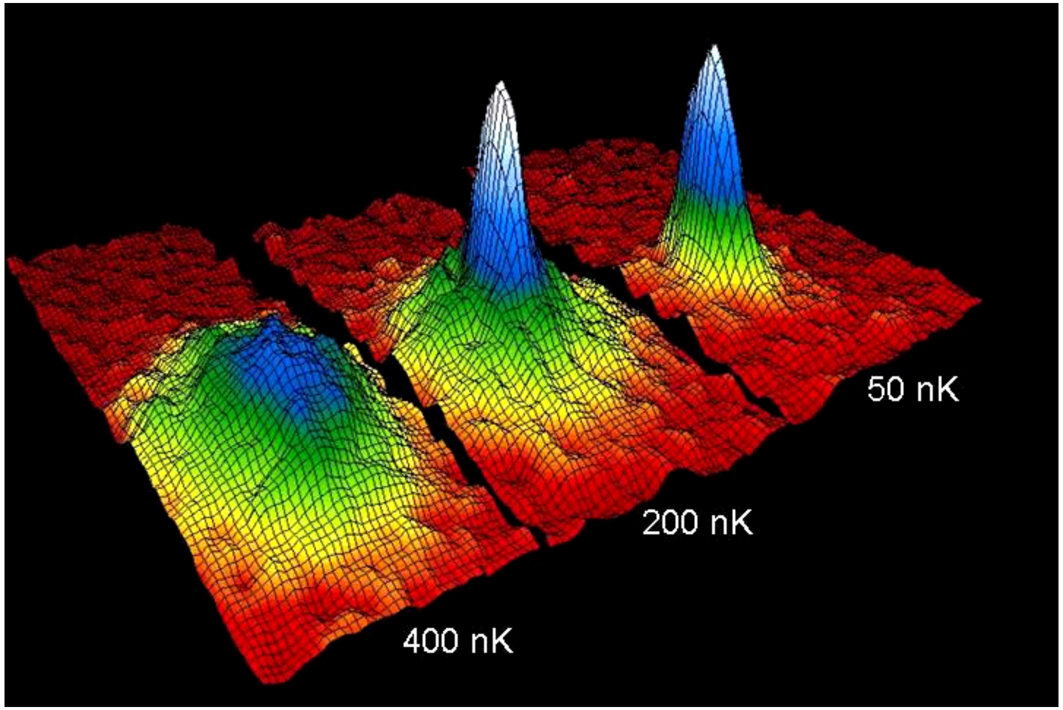
Bose-Einstein Condensation in a Gas of Sodium Atoms

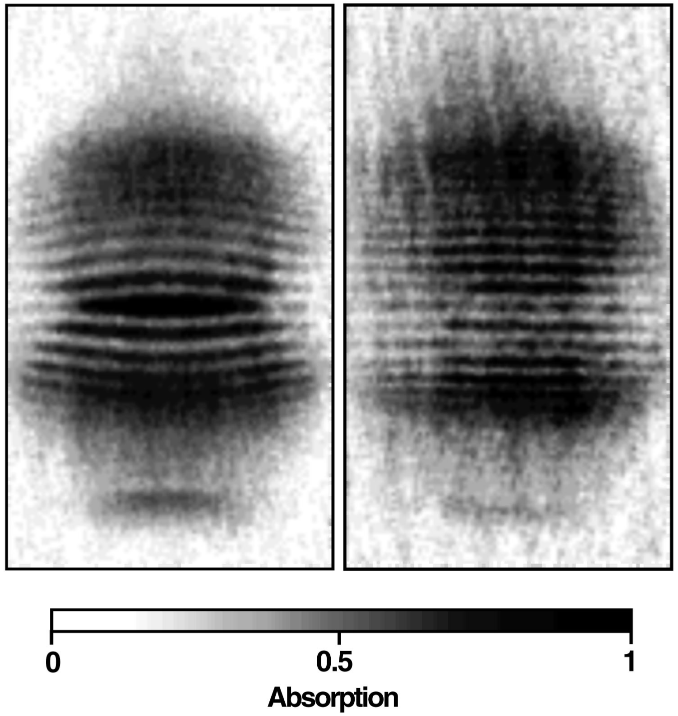
K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle
Department of Physics and Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 17 October 1995)

- Bose-Einstein Condensation in a Gas of Sodium Atoms, K.B. Davis, M.O. Mewes, M.R. Andrews, N.J. van Druten, D.S. Durfee, D.M. Kurn, and W. Ketterle, Phys. Rev. Lett. 75, 3969 (1995).

Striking properties of BECs



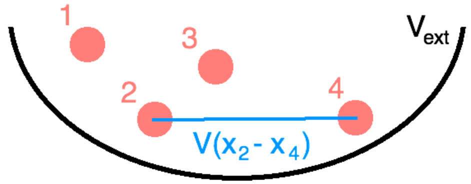
Momentum distribution of the rubidium atoms at different temperatures



Interference between two expanding sodium BECs

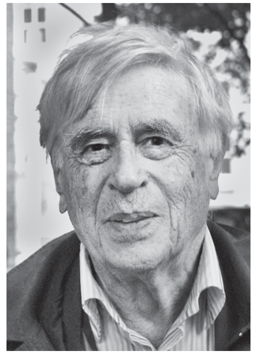
The Gross-Pitaevskii model

$$\hat{H} = \int \hat{\Psi}^\dagger(\mathbf{x}_1)\left[\frac{\hat{p}^2}{2m} + V_{ext}(\mathbf{x}_1)\right]\hat{\Psi}(\mathbf{x}_1)d\mathbf{x}_1$$
$$+ \frac{1}{2}\int \hat{\Psi}^\dagger(\mathbf{x}_1)\hat{\Psi}^\dagger(\mathbf{x}_2)V(\mathbf{x}_1 - \mathbf{x}_2)\hat{\Psi}(\mathbf{x}_2)\hat{\Psi}(\mathbf{x}_1)d\mathbf{x}_{12}$$
$$+ \dots$$



classical analogue

$$H = \sum_i \left[\frac{p_i^2}{2m} + V_{ext}(x_i)\right] + \frac{1}{2}\sum_{i,j} V(x_i - x_j)$$



- $\hat{\Psi}(\mathbf{x}) \simeq \psi(\mathbf{x})$, the system is very cold and highly occupied
- $V(\mathbf{x}_1 - \mathbf{x}_2) = \frac{4\pi\hbar^2 a}{m}\delta(\mathbf{x}_1 - \mathbf{x}_2)$, only point-like binary collisions

The evolution of the condensate wave function is then given by computing the Heisenberg evolution

$i\hbar\partial_t\hat{\Psi} = [\hat{\Psi}, \hat{H}]$ leads to Gross-Pitaevskii equation (GPE)

$$i\hbar\partial_t\psi(\mathbf{x},t) = \left(-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\mathbf{x}) + \frac{4\pi\hbar^2 a}{m}|\psi(\mathbf{x},t)|^2\right)\psi(\mathbf{x},t)$$

The dimensionless formulation of GP

$$i\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\nabla^2\psi - g|\psi|^2\psi = V\psi, \quad \text{with} \quad g = \frac{4\pi\hbar^2 a_s}{m}$$

$$\text{Setting} \quad V \equiv 0, \quad \psi \to \sqrt{\rho_\infty}\,\psi, \quad t \to \frac{\hbar}{g\rho_\infty}\,t, \quad x \to \xi\, x$$

$$i\partial_t\psi + \left(\frac{\hbar^2}{2mg\rho_\infty}\,\xi^{-2}\right)\nabla^2\psi - |\psi|^2\psi = 0 \quad \Longrightarrow \quad \xi = \frac{\hbar}{\sqrt{2mg\rho_\infty}}$$

Thus for an infinite BEC there exist a length scale inversely proportional to the square root of the density ρ_∞ that is called healing length ξ

$$i\partial_t\psi + \nabla^2\psi - |\psi|^2\psi = 0$$

The resulting GP is nothing but the celebrated nonlinear Schroedinger equation in three dimensions

Conserved quantities, V=0 for simplicity

Under appropriate boundary conditions (infinite volume or periodic box) the GP model conserves:

the mass $$M = \int_V |\psi|^2 dV$$

the energy $$E = \int_V \left(|\nabla\psi|^2 + \frac{1}{2}|\psi|^4 \right) dV$$

linear momentum $$\mathbf{P} = -2i \int_V \left(\psi^* \nabla\psi - \psi\nabla\psi^*\right) dV$$

and angular momentum $$\mathbf{M}_{\mathbf{x}'} = -2i \int_V \mathbf{x}' \times \left(\psi^* \nabla\psi - \psi\nabla\psi^*\right) dV$$

GP as a model for superfluids

$$i\partial_t\psi + \nabla^2\psi - |\psi|^2\psi = 0$$

In order to prove that the GP above actually describes a superfluid we can introduce the Madelung's transformation

$$\psi(\mathbf{x}, t) = \sqrt{\rho(\mathbf{x}, t)}\, e^{i\theta(\mathbf{x},t)}$$

and notice that after defining a velocity field as $\mathbf{v} = 2\nabla\theta$ and splitting the into real and imaginary part, the GP equation results in

$$\begin{cases} \dfrac{\partial\rho}{\partial t} + \nabla\cdot(\rho\,\mathbf{v}) = 0 \\ \rho\left(\dfrac{\partial v_j}{\partial t} + v_k\dfrac{\partial v_j}{\partial x_k}\right) = -\dfrac{\partial p}{\partial x_j} + \dfrac{\partial \Sigma_{jk}}{\partial x_k} \end{cases}$$

$p = \rho^2$ is a pressure term, $\Sigma_{jk} = \rho\,\frac{\partial^2(\ln\rho)}{\partial x_j\partial x_k}$ is the quatum stress tensor

GP as a model for superfluids

In other words the GP equation can be mapped into an equation for the density field ρ and velocity field $\mathbf{v} = 2\nabla\theta$ that satisfy

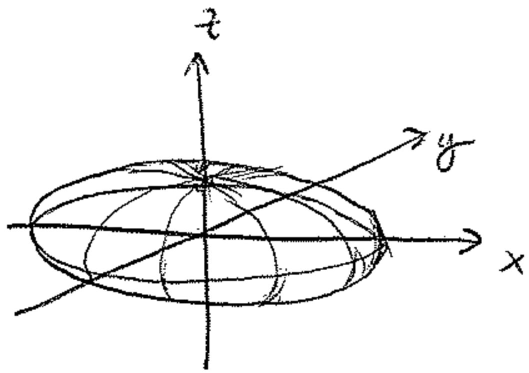
$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho\,\mathbf{v}) = 0 \\ \rho\left(\dfrac{\partial v_j}{\partial t} + v_k \dfrac{\partial v_j}{\partial x_k}\right) = -\dfrac{\partial p}{\partial x_j} + \dfrac{\partial \Sigma_{jk}}{\partial x_k} \end{cases}$$

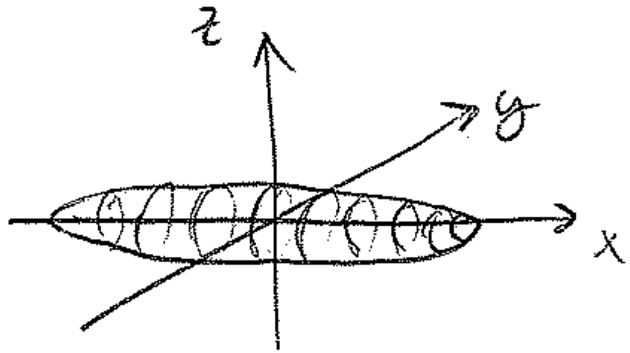
that **describes an inviscid, irrotational, barotropic fluid**

^{4}He	Alkali BECs
$\xi \sim \mathring{A}, \quad L/\xi \simeq 10^4 - 10^5$	$\xi(m, a_s, \rho_\infty), \quad L/\xi \simeq 1 - 10^2$
Only qualitative model for liquid helium, too rarefied model!	Very good model when $T \simeq 0$

BECs in potential traps

BECs are trapped in harmonic potentials, and by increasing the potential frequencies, one can reduce their dimensions:





2-d, **“pancake” condensate** 1-d, **“cigar” condensate**

Recently an almost perfect box container with a step potential in each direction was created

PRL **110,** 200406 (2013) PHYSICAL REVIEW LETTERS week ending 17 MAY 2013

Bose-Einstein Condensation of Atoms in a Uniform Potential

Alexander L. Gaunt, Tobias F. Schmidutz, Igor Gotlibovych, Robert P. Smith, and Zoran Hadzibabic
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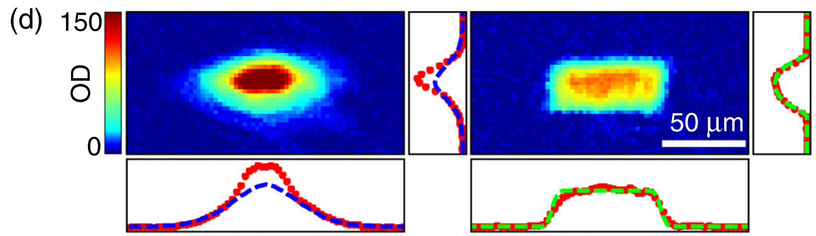
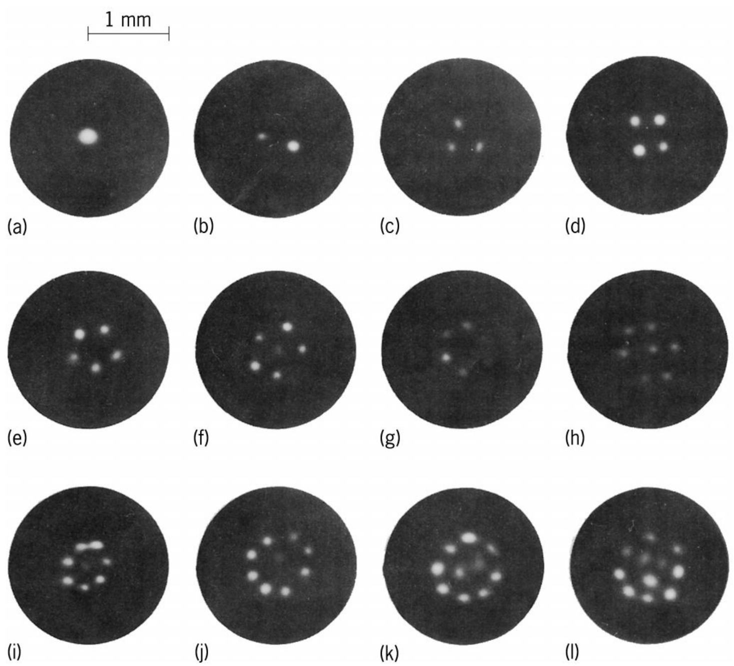
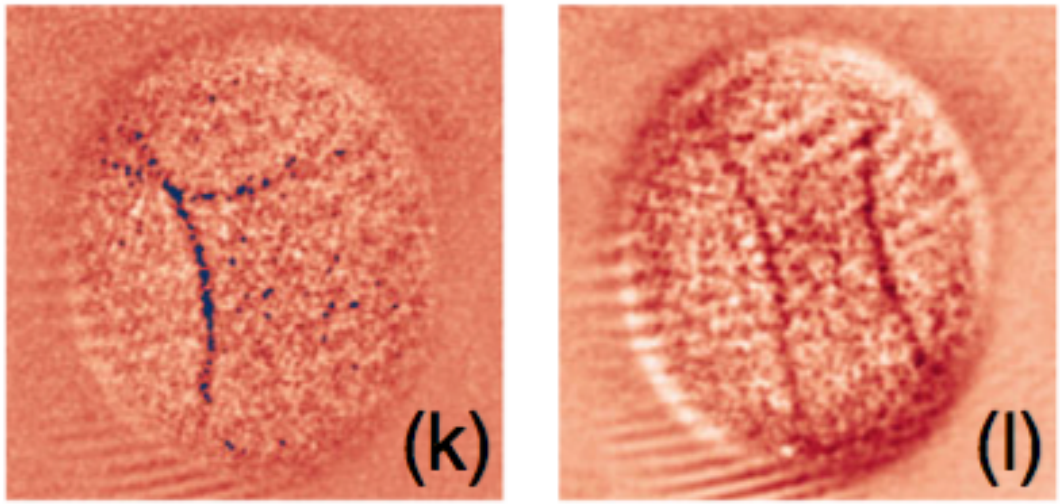
(Received 14 March 2013; published 16 May 2013)



Weakly nonlinear waves and coherent structures in the Gross-Pitaevskii model





Dr Davide Proment, University of East Anglia (UEA)
Torino, 17th November 2020

de Broglie limit, 4-wave interactions

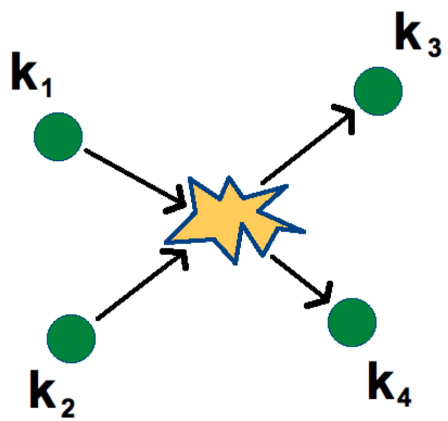
Let us work with the standard GP form in an unbounded domain

$$i\partial_t \psi + \beta \nabla^2 \psi + \alpha |\psi|^2 \psi = 0$$

In the limit of small nonlinearity (vanishing density or scales much smaller than the healing length), a weak wave turbulence closure can be applied, leading to the 4-wave kinetic equation

$$\partial_t n_1 = (4\pi\alpha)^2 \int n_1 n_2 n_3 n_4 \left(\frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4} \right) \delta(\mathbf{k}_{12}^{34})\, \delta(\omega_{12}^{34})\, d\mathbf{k}_{234}$$

where $\omega_{\mathbf{k}} = \beta |\mathbf{k}|^2$



- Only resonant interactions contribute

$$\begin{cases} \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4 \\ \omega_1 + \omega_2 = \omega_3 + \omega_4 \end{cases}$$

- Irreversibility in the dynamics

$$S(t) = \int \log n_{\mathbf{k}}\, d\mathbf{k}\,, \quad \dot{S}(t) \geq 0$$

de Broglie limit, Rayleigh-Jean

It is a quick exercise to show that the equilibrium distribution results in

$$n_{RJ}(\mathbf{k}) = \frac{T}{\mu + \Pi \cdot \mathbf{k} + \omega(\mathbf{k})}$$

That is nothing but the Rayleigh-Jean equilibrium. By introducing a small-scale cut-off to heal the ultraviolet catastrophe one finds that in 3D, the extensive quantities per unit of volume are related to the intensive ones as [Connaughton et al., PRL 95, 2005]

$$\frac{N}{V}^{(3D)} = \int_0^{k_c} \int_\Omega \frac{T}{\mu + \beta k^2}\, k^2 d\Omega dk = \frac{4\pi T k_c}{\beta} \left[1 - \sqrt{\frac{\mu}{\beta k_c^2}} \arctan\left(\sqrt{\frac{\beta k_c^2}{\mu}}\right)\right]$$

$$\frac{E}{V}^{(3D)} = \int_0^{k_c} \int_\Omega \beta k_c^2 \frac{T}{\mu + \beta k^2}\, k^2 d\Omega dk = \frac{4\pi T k_c^3}{3\beta} \left[1 - \frac{3\mu}{\beta k_c^2} + \frac{3\mu^{3/2}}{\beta^{3/2} k_c^3} \arctan\left(\sqrt{\frac{\beta k_c^2}{\mu}}\right)\right]$$

In 3D **a spontaneous condensation** arises below a non-zero temperature (or above a non-zero density) when the energy per mode is below $E/N^{(3D)} = \beta k_c^2/3$.

Numerical studies of spontaneous condensation in 3D

PHYSICAL REVIEW A **66**, 013603 (2002)

Scenario of strongly nonequilibrated Bose-Einstein condensation

Natalia G. Berloff[1,*] and Boris V. Svistunov[2,†]

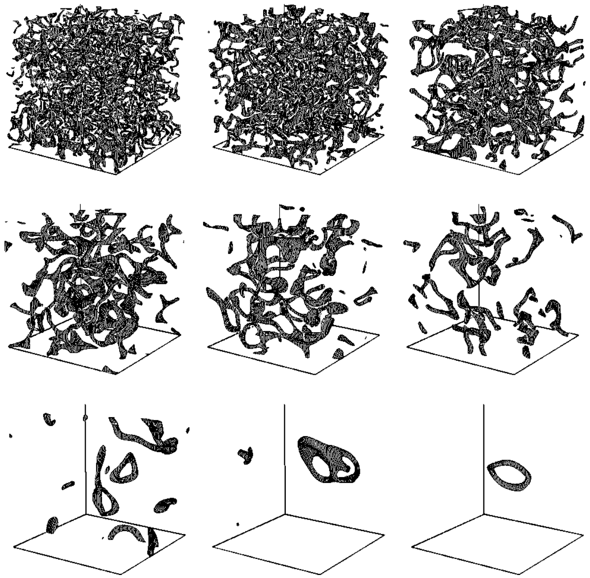


FIG. 4. Evolution of topological defects in the phase of the long-wavelength part $\tilde{\psi}$ of the field ψ in the computational box 128^3. The defects are visualized by isosurfaces $|\tilde{\psi}|^2 = 0.05\langle|\tilde{\psi}|^2\rangle$. High-frequency spatial waves are suppressed by the factor $\max\{1 - k^2/k_c^2, 0\}$, where the cutoff wave number is chosen according to the phenomenological formula $k_c = 9 - t/1000$.

PRL **95**, 263901 (2005) PHYSICAL REVIEW LETTERS week ending 31 DECEMBER 2005

Condensation of Classical Nonlinear Waves

Colm Connaughton,[1] Christophe Josserand,[2] Antonio Picozzi,[3] Yves Pomeau,[1] and Sergio Rica[4,1]

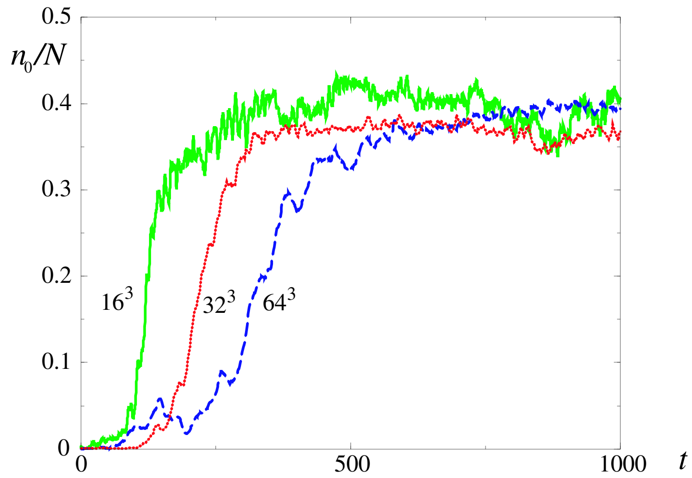


FIG. 1 (color online). Numerical simulations of the NLS Eq. (1) showing the temporal evolution of condensed particles n_0/N in 3D: independently of the number of computational modes, n_0/N tends to converge for long interaction times ($\langle H\rangle/V = 2$, $N/V = 1/2$, and $k_c = \pi/dx$ where $dx = 1$ refers to the spatial discretization of the NLS equation).

To fully describe the phase transition one cannot use WT and should refer to the standard works in statistical mechanics on the ϕ^4 model!

The weakly-nonlinear Bogoliubov limit

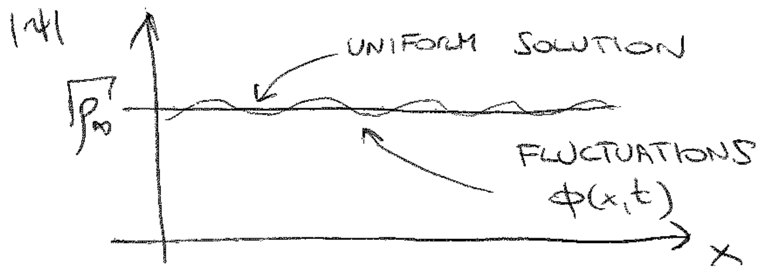
We have seen that one can consider infinitesimal Bogoliubov fluctuations on top of a uniform condensate solution

$$\psi(\mathbf{x},t) = \left[\sqrt{\rho_\infty} + \epsilon\,\phi(\mathbf{x},t)\right] e^{-i\mu t}\,, \ \ \text{with}\ \ \epsilon \ll \sqrt{\rho_\infty}$$

Because these are small, a WT closure can be applied to the distribution of the fluctuations, resulting in the 3-wave kinetic equation

[Proment et al., PRA 90, 2014]

$$\partial_t n_{\mathbf{1}} = \int \left(\mathcal{R}^1_{2,3} - \mathcal{R}^3_{1,2} - \mathcal{R}^2_{1,3}\right) d\mathbf{k}_{23}$$



where $\mathcal{R}^1_{2,3} = |V^1_{2,3}|^2\,(n_2 n_3 - n_1 n_2 - n_1 n_3)\,\delta(\mathbf{k}^1_{2,3})\delta(\omega^1_{2,3})$

$$\omega_{Bog}(\mathbf{k}) = \pm\sqrt{\beta}\,|\,\mathbf{k}\,|\sqrt{\beta\,|\,\mathbf{k}^2 - 2\alpha\beta\rho_\infty} \quad \text{and} \quad \mu = -\,\alpha\rho_\infty$$

(Bogoliubov dispersion relation)

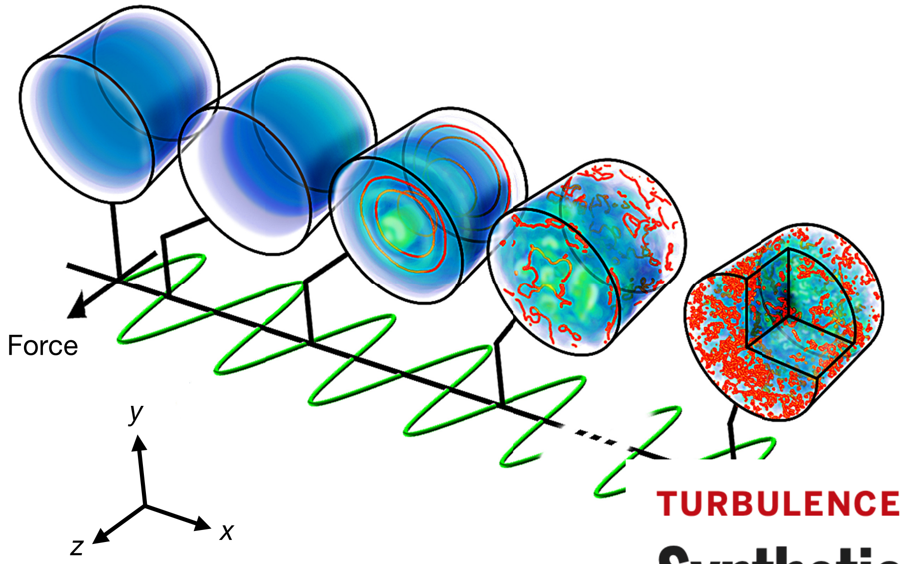
Wave turbulence in BECs?

LETTER
doi:10.1038/nature20114

Emergence of a turbulent cascade in a quantum gas

Nir Navon[1], Alexander L. Gaunt[1], Robert P. Smith[1] & Zoran Hadzibabic[1]

a



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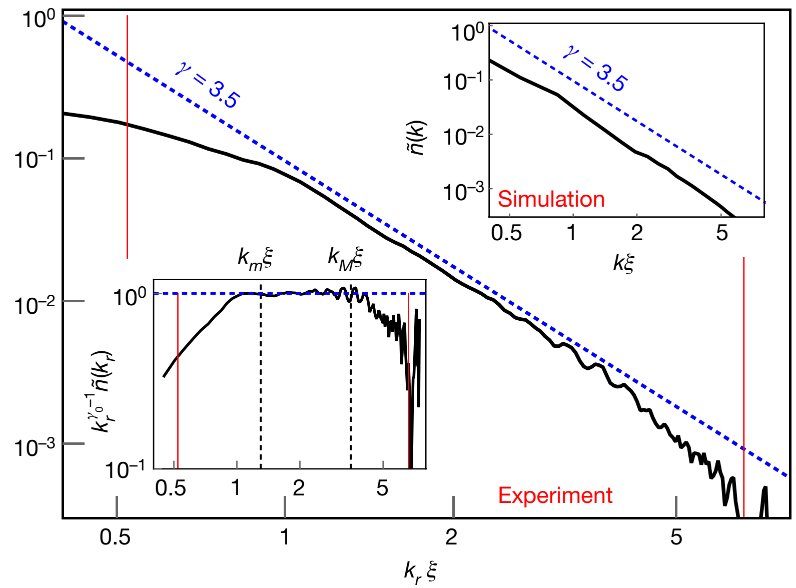


TURBULENCE

Synthetic dissipation and cascade fluxes in a turbulent quantum gas

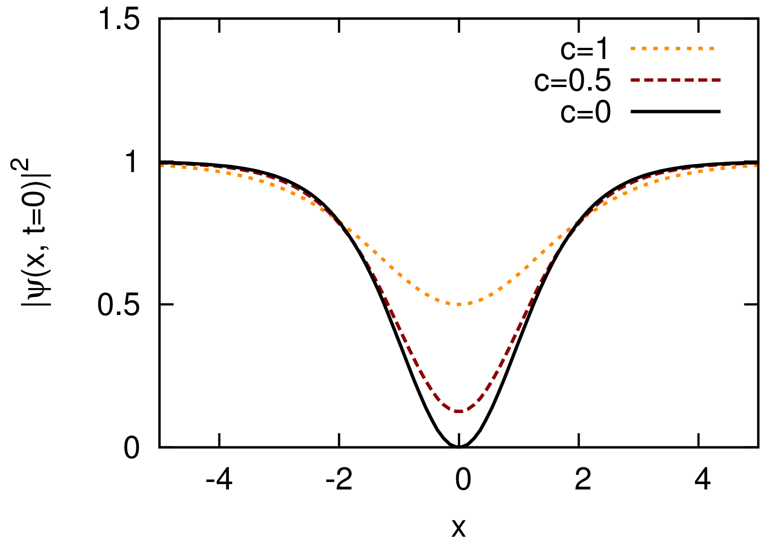
Nir Navon[1]*, Christoph Eigen[2], Jinyi Zhang[2], Raphael Lopes[2]†, Alexander L. Gaunt[2,3], Kazuya Fujimoto[4]*‡, Makoto Tsubota[5], Robert P. Smith[2,6], Zoran Hadzibabic[2]

Grey solitons in the 1D GP model

Usually the bosons in the superfluid state self-interact via a repulsive potential. This results in $\alpha < 0\,,\;\; \beta > 0$ in GP. In 1D the model is nothing but the nonlinear defocusing Schroedinger equation

$$i\partial_t\psi + \beta\partial_{xx}\psi + \alpha|\psi|^2\psi = 0$$

If one seeks for travelling wave solutions depending on the unique dependent variable $\zeta = x - ct$, given c the wave velocity, one obtain the celebrated grey soliton solution over the uniform density at infinity



[here $\alpha = -1\,,\;\; \beta = 1$ and $\rho_\infty = 1$]

$$\psi(x,t) = \left\{\sqrt{\frac{2-c^2}{2}}\tanh\left[\frac{\sqrt{2-c^2}}{2}(x-ct)\right] + i\frac{c}{\sqrt{2}}\right\}\exp(-it)$$

Experimental evidence of grey solitons

VOLUME 83, NUMBER 25 PHYSICAL REVIEW LETTERS 20 DECEMBER 1999

Dark Solitons in Bose-Einstein Condensates

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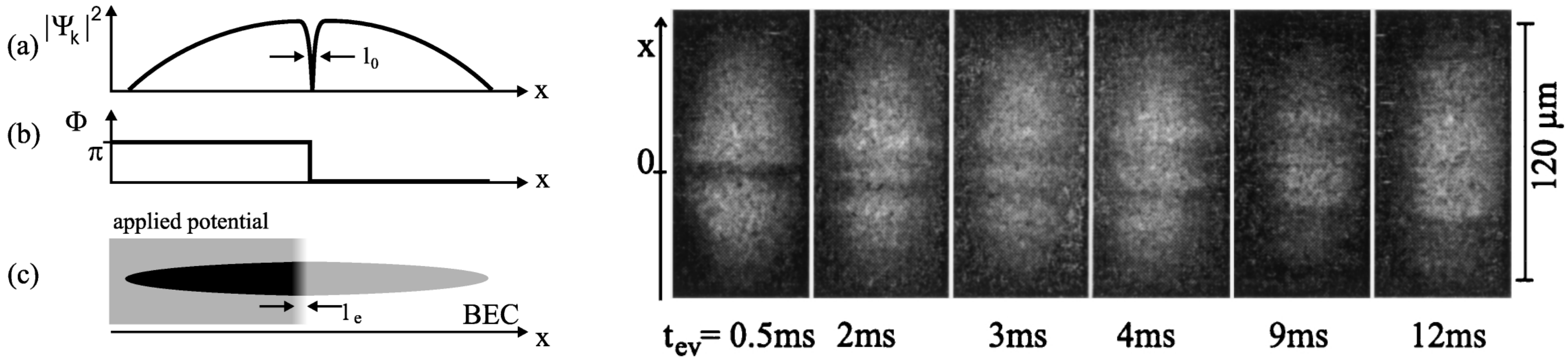
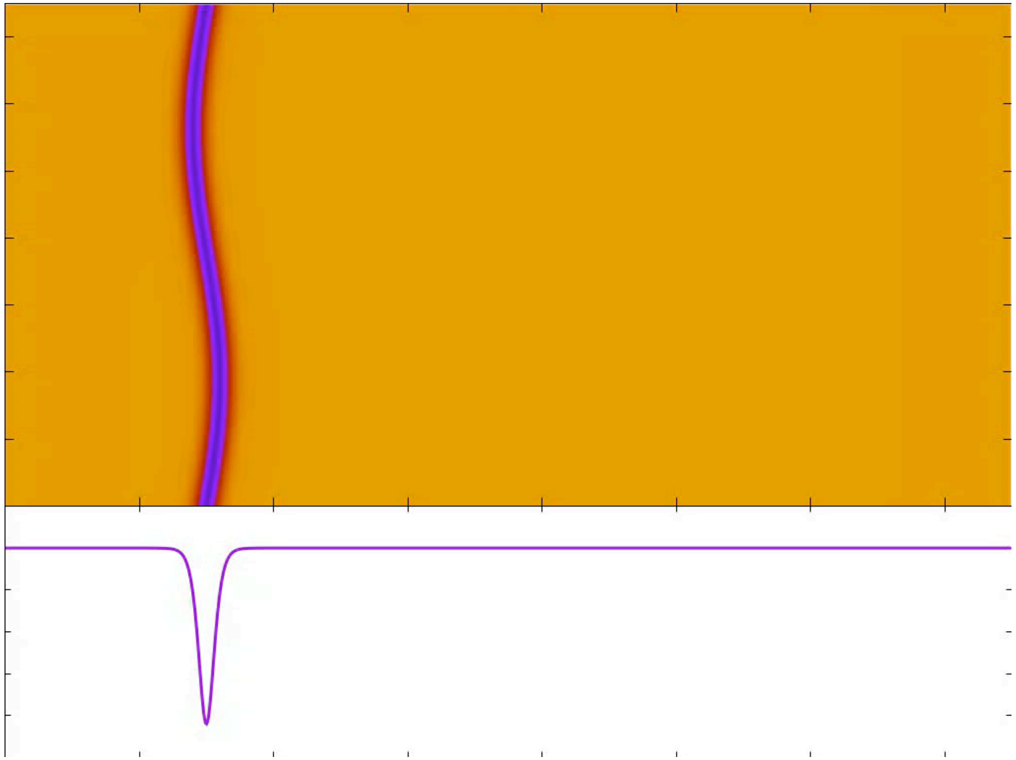
(Received 27 October 1999)



Instability and decay of a grey soliton

Let us come back to the defocusing (repulsive boson interaction) GP model, the only case that do not exhibit collapses **in more than 1D**. The evolution of a grey soliton (in the x-direction) subjected to a transverse perturbation (in the y-direction) results as follows



[Top: the density (yellow is high density and dark blue is low density of the two dimensional filed. Bottom: one-dimensional evolution of the grey soliton profile for comparison]

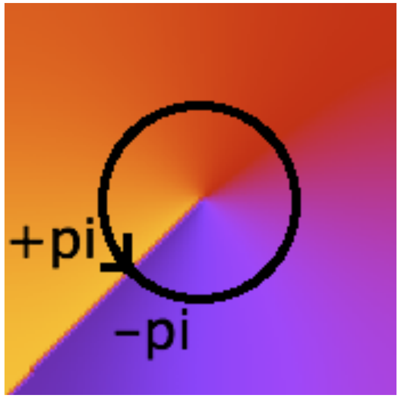
The soliton is unstable and decay producing other structures that carry the linear momentum of the initial solution. Due to the dynamics of the decay, this mechanisms is called “snake instability”.

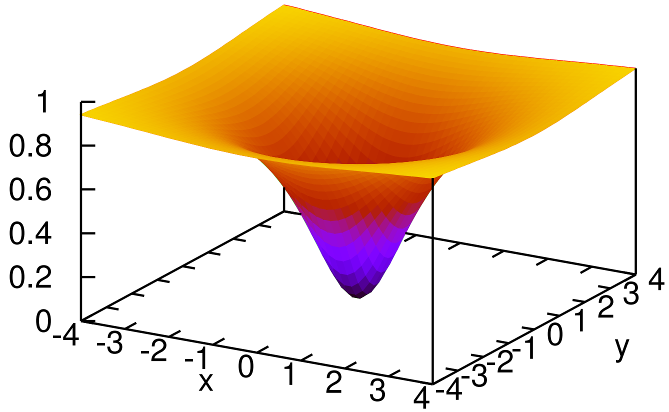
Quantised vortices in the GP model

Even if the velocity field of the superfluid $\mathbf{v} = 2\beta\nabla\phi$ obtained by applying the Madelung transformation $\psi = \sqrt{\rho}\exp{(i\phi)}$ is formally irrotational, vortices may appear in the system as topological defects of the phase field.

Let us assume that the phase field changes as

$$\Delta\phi = n\,2\pi\,, \quad n \in \mathbb{Z}$$





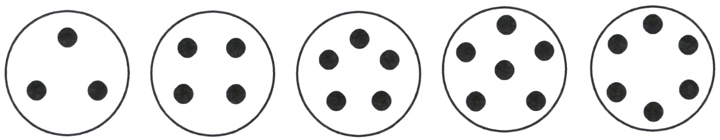
Then, the order parameter still remains single-valued but the velocity circulation results quantised as

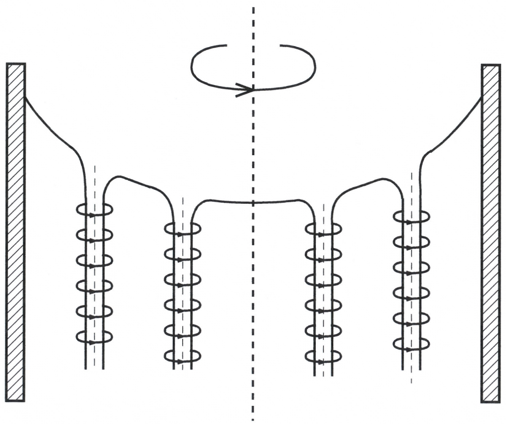
$$\Gamma = \oint_{\mathcal{C}} \mathbf{v}\cdot d\mathbf{l} = 2\beta\oint_{\mathcal{C}} \nabla\phi\cdot d\mathbf{l} = n\,4\pi\beta = n\,\kappa$$

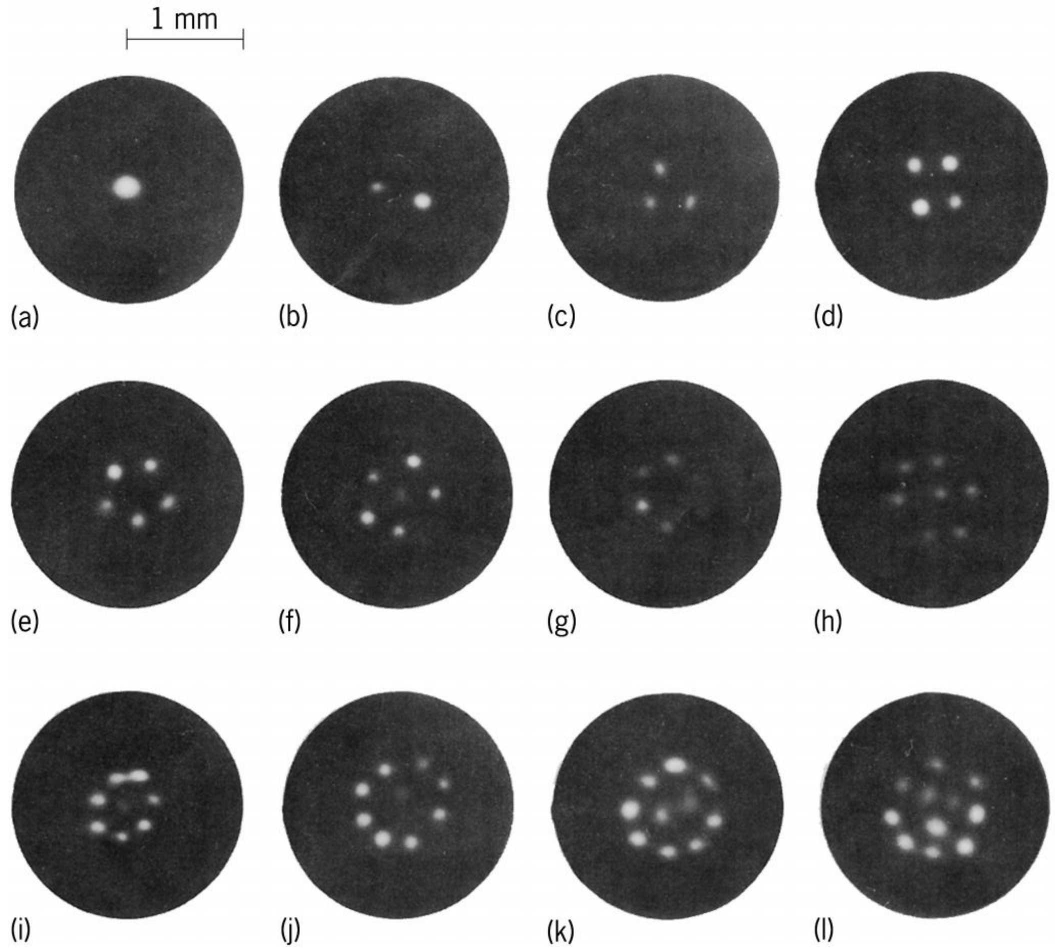
The physical value $\kappa = 4\pi\beta$ is usually called the Feynman—Onsager **quantum of circulation**.

Quantum vortices in helium-4

What does happen if we **rotate the liquid helium container** and then lower the temperature below the superfluid transition temperature?





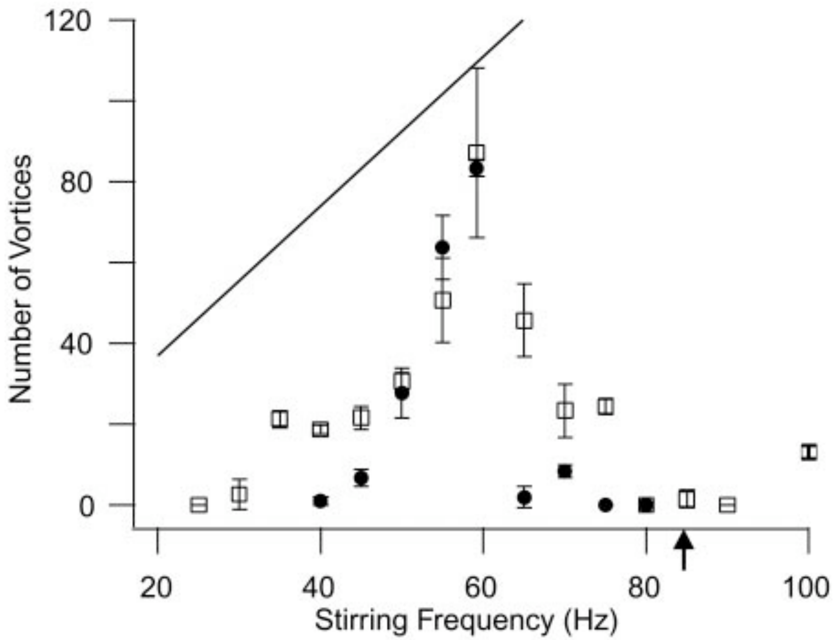


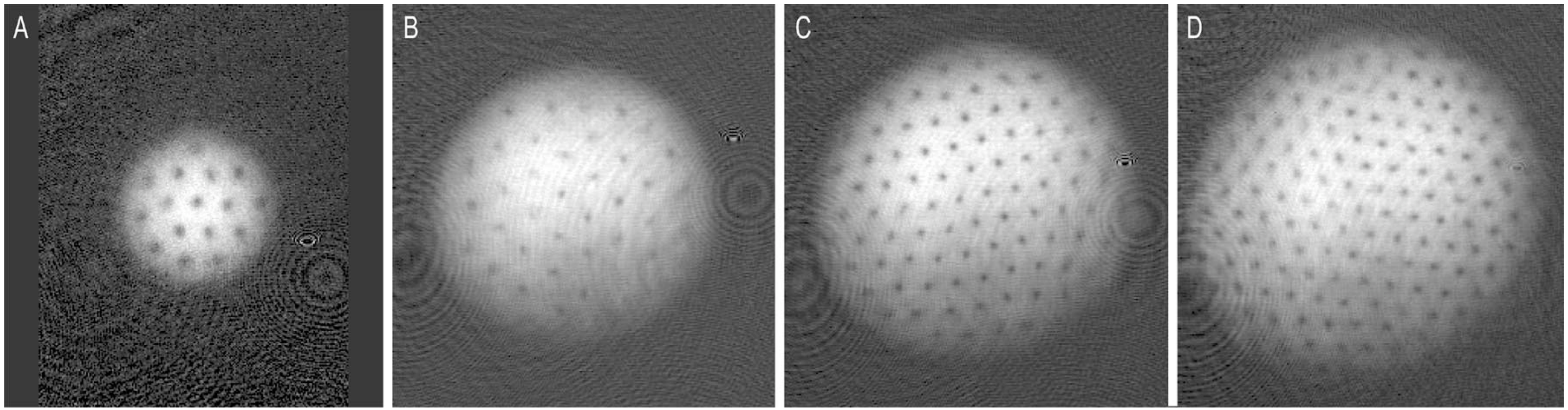
[Yarmchuk et al, PRL 43, 1979]

Quantum vortices in BECs

A similar effect can be obtained
in a BEC by stirring it with a moving laser acting like a "spoon".

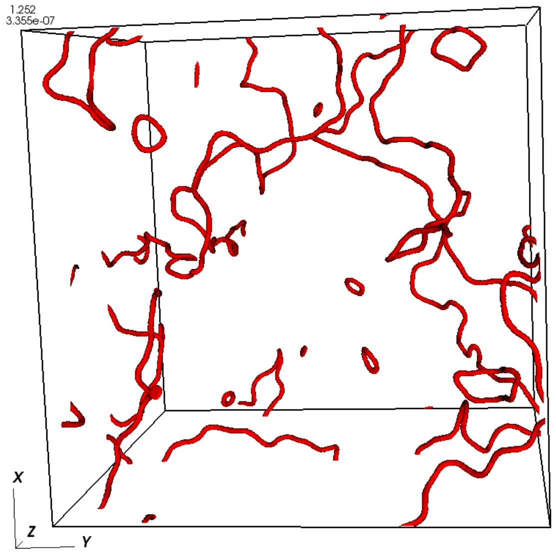
[Abo-Shaee et al, Science 292, 2001]



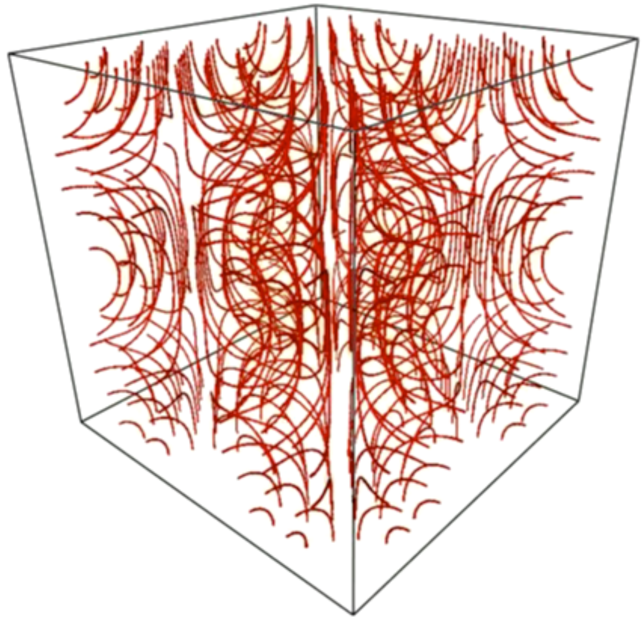


Quantised vortices in 3D GP model

In 3D quantised vortices form closed loops or lines that must end at the boundaries.

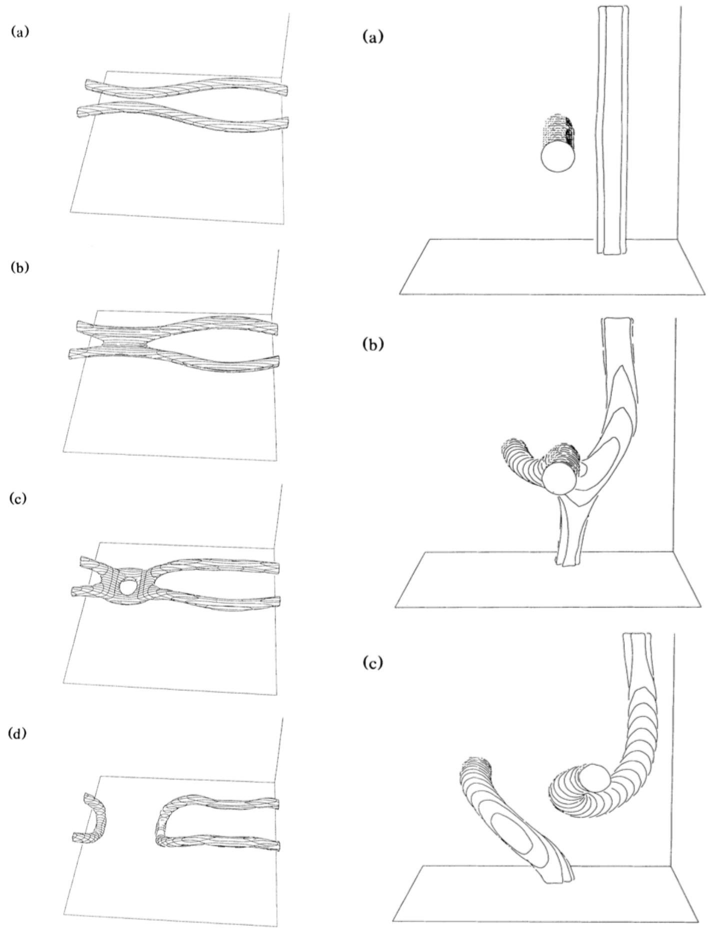


- vortices self-induce their motion
- vortices can reconnect
- the filaments carry helicoidal excitations called Kelvin waves
- they interact with density excitations (sound)



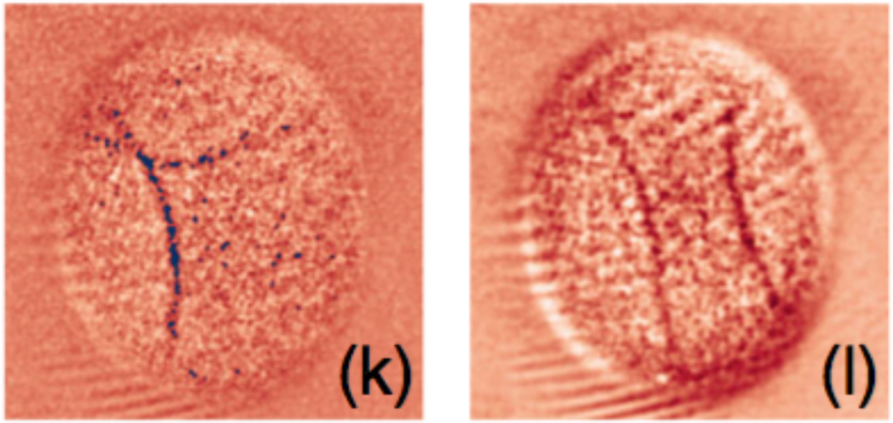
All these features are the core of **quantum turbulence**.

Quantum vortex reconnections



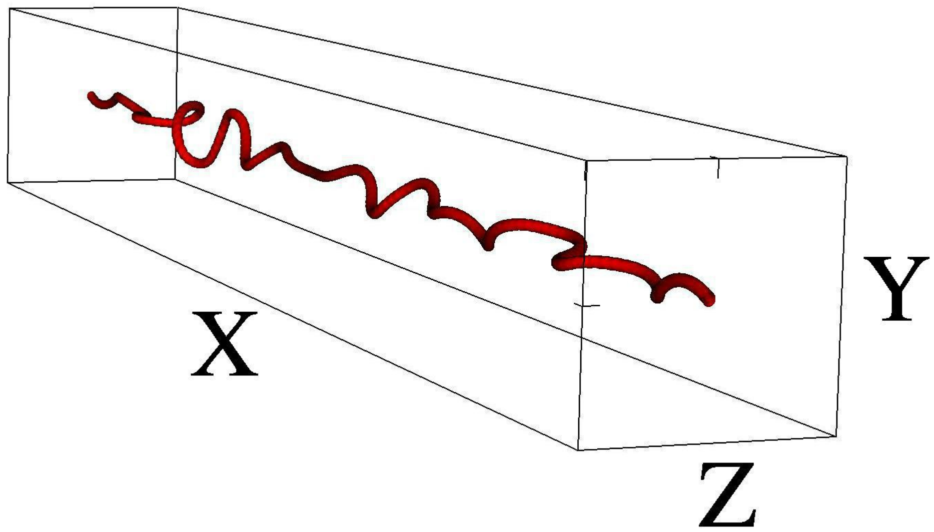
[Koplik & Levine, PRL 71, 1993]

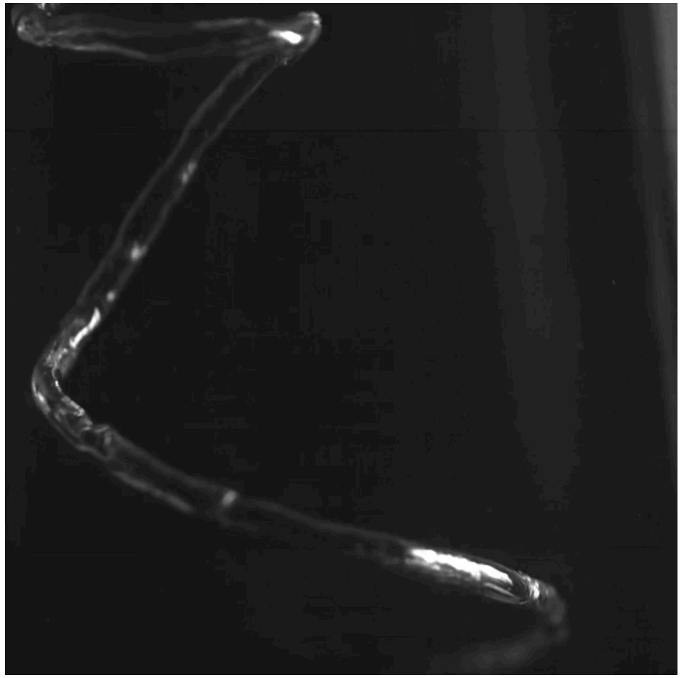
[Paoletti et al., PRL 101, 2008]



[Serafini et al., PRL 115, 2015]

Kelvin waves on vortex lines





Given the vortex position at $x = const.$, the complex Kelvin wave variable w at a time t results in

By analogy, see an experiment showing Kelvin waves and vortices in water [Tsoy et al, JoP CS 980, 2018]

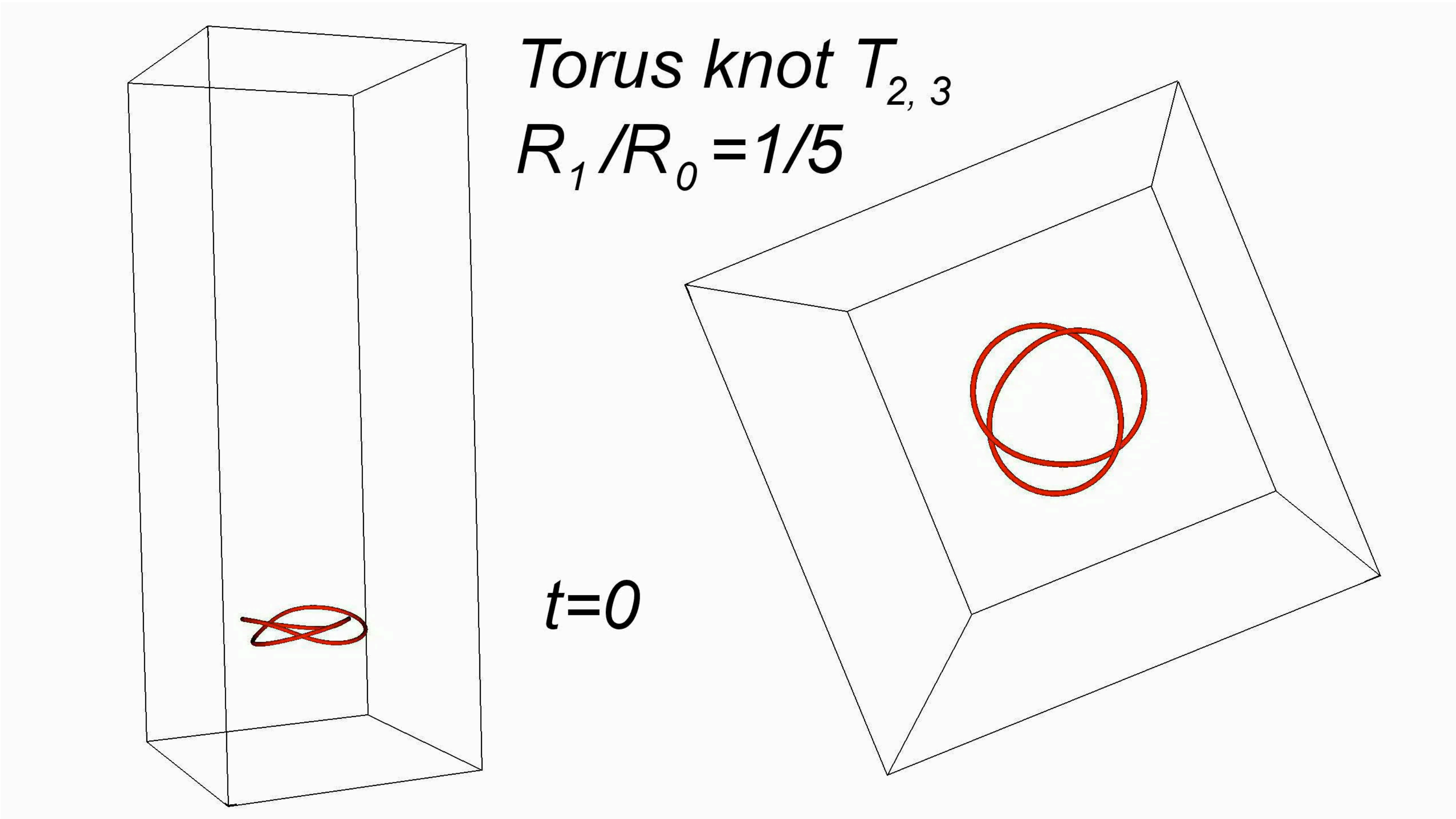
$$w(x,t) = y_v(x,t) + iz_v(x,t) = \int \tilde{w}(k,t) \exp{(ikx)}\, dk$$

Example of a quantised vortex knot

PHYSICAL REVIEW E **85**, 036306 (2012)

Vortex knots in a Bose-Einstein condensate

Davide Proment,[1,2,*] Miguel Onorato,[1,2] and Carlo F. Barenghi[3]



Moving obstacles in a superfluid

PHYSICAL REVIEW LETTERS **123,** 154502 (2019)

Starting Flow Past an Airfoil and its Acquired Lift in a Superfluid

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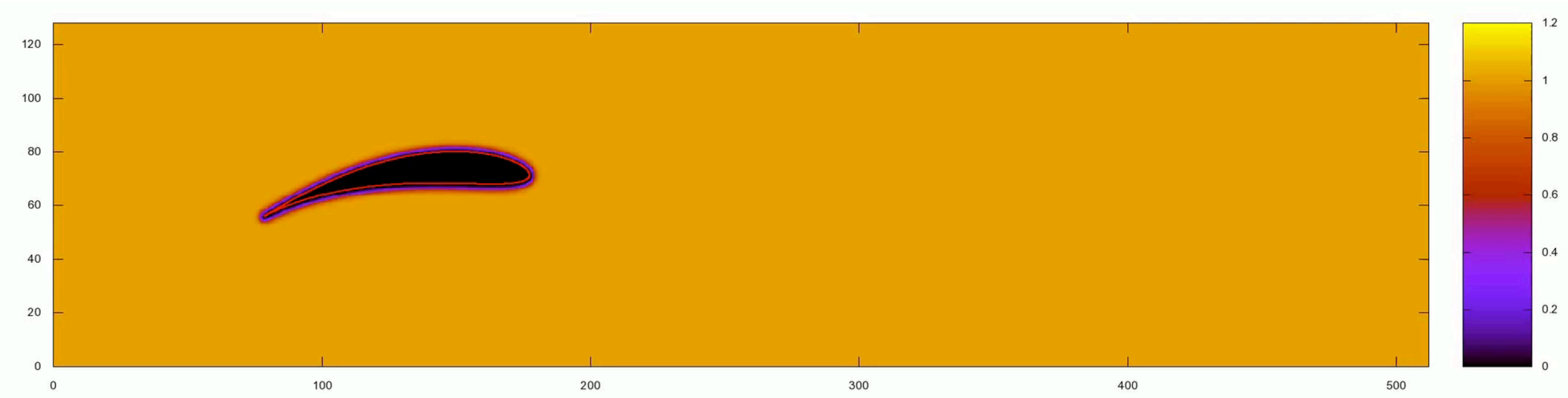

Other projects with the GP model

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IOP Publishing
Journal of Physics A: Mathematical and Theoretical
J. Phys. A: Math. Theor. **52** (2019) 305501 (11pp)
https://doi.org/10.1088/1751-8121/ab2607

Clustering and phase transitions in a 2D superfluid with immiscible active impurities

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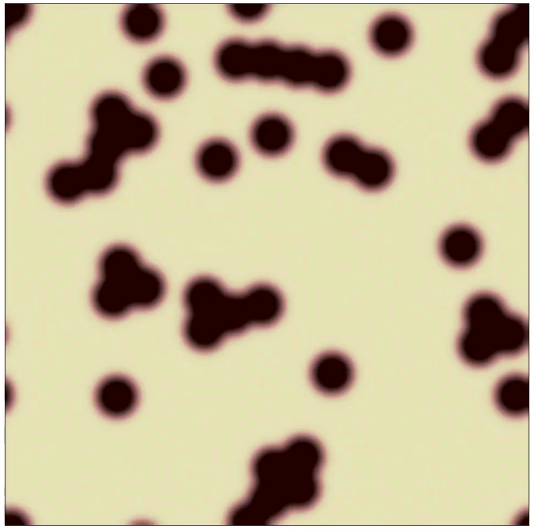

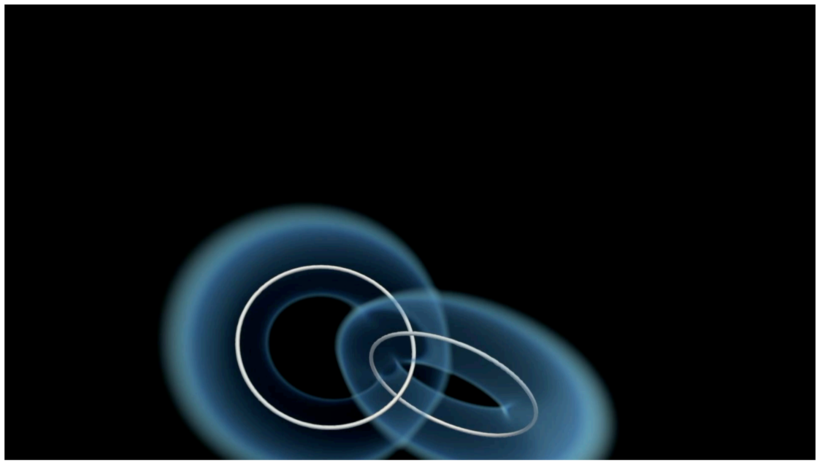

PHYSICAL REVIEW LETTERS **125,** 164501 (2020)

Irreversible Dynamics of Vortex Reconnections in Quantum Fluids

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Madelung's transformations

Start with $V \equiv 0$, unbounded domain

$$i\partial_t \psi + \beta \bar{\nabla}^2 \psi + \alpha |\psi|^2 \psi = 0 \quad , \quad \psi : \mathbb{R}^d \to \mathbb{C}$$

and take $\psi = \sqrt{\rho}\, e^{i\phi}$, $\rho, \phi : \mathbb{R}^d \to \mathbb{R}$

then the differential operations results in

$$\partial_t \psi = \frac{1}{2\sqrt{\rho}} \partial_t \rho\, e^{i\phi} + i\sqrt{\rho}\, \partial_t \phi\, e^{i\phi} = \left(\frac{1}{2\sqrt{\rho}} \partial_t \rho + i\sqrt{\rho}\, \partial_t \phi \right) e^{i\phi}$$

$$\bar{\nabla}^2 \psi = (\bar{\nabla} \cdot \bar{\nabla}) \psi = \left(\bar{\nabla}^2 \sqrt{\rho} \right) e^{i\phi} + 2 \left(\bar{\nabla} \sqrt{\rho} \right) \cdot \left(\bar{\nabla} e^{i\phi} \right) + \sqrt{\rho} \left(\bar{\nabla}^2 e^{i\phi} \right)$$

as $\nabla^2 e^{i\phi} = \bar{\nabla} \cdot (i \bar{\nabla}\phi\, e^{i\phi}) = i \bar{\nabla}^2 \phi\, e^{i\phi} - (\bar{\nabla}\phi)^2 e^{i\phi}$

$$= \left(\bar{\nabla}^2 \sqrt{\rho} - \sqrt{\rho} (\nabla \phi)^2 + 2i\, \bar{\nabla} \sqrt{\rho} \cdot \bar{\nabla} \phi + i \sqrt{\rho}\, \bar{\nabla}^2 \phi \right) e^{i\phi}$$

The imaginary part of GP is

$$\frac{1}{2\sqrt{\rho}} \partial_t \rho\, \cancel{e^{i\phi}} + \frac{\beta}{\sqrt{\rho}} \bar{\nabla} \rho \cdot \bar{\nabla} \phi\, \cancel{e^{i\phi}} + \beta \sqrt{\rho}\, \bar{\nabla}^2 \phi\, \cancel{e^{i\phi}} = 0$$

assuming $\sqrt{\rho} \neq 0$ (that must be true otherwise the transformation is not valid)

$$\partial_t \rho + 2\beta \bar{\nabla} \rho \cdot \bar{\nabla} \phi + 2\beta \rho \bar{\nabla}^2 \phi = 0$$

$$\partial_t \rho + \bar{\nabla} \left(\rho\, 2\beta \bar{\nabla} \phi \right)$$, thus by setting $\bar{v} = 2\beta \bar{\nabla} \phi$

one obtains $\partial_t \rho + \bar{\nabla} (\rho \bar{v}) = 0$ which is the continuity equation for a compressible fluid having density ρ and velocity field $\bar{v}$.

The real part of GP is

$$-\sqrt{\rho}\,\partial_t \phi\, \cancel{e^{i\phi}} + \beta \nabla^2\sqrt{\rho}\, \cancel{e^{i\phi}} - \beta\sqrt{\rho}(\bar{\nabla}\phi)^2 \cancel{e^{i\phi}} + \alpha\rho\sqrt{\rho}\,\cancel{e^{i\phi}} = 0$$

$$-\partial_t\phi + \beta\,\frac{\bar{\nabla}^2\sqrt{\rho}}{\sqrt{\rho}} - \beta(\bar{\nabla}\phi)^2 + \alpha\rho = 0$$

We want to express also this equation in terms of $\bar{v} = 2\beta\bar{\nabla}\phi$, hence we will take the gradient of it and multiply by 2β to get

$$-\partial_t\bar{v} + 4\beta^2\,\bar{\nabla}\left(\frac{\bar{\nabla}^2\sqrt{\rho}}{\sqrt{\rho}}\right) - \frac{1}{2}\bar{\nabla}\left(2\beta\bar{\nabla}\phi\right)^2 + 2\alpha\beta\,\bar{\nabla}\rho = 0$$

Using the vectorial identity

$$\bar{\nabla}(\bar{A}\cdot\bar{A}) = 2(\bar{A}\cdot\bar{\nabla})\bar{A} + 2\bar{A}\times(\bar{\nabla}\times\bar{A})$$

and the fact that $\bar{v}$ is irrotational $(\bar{\nabla}\times\bar{v}=0)$ because it is the gradient of a scalar, we obtain

$$\partial_t\bar{v} + (\bar{v}\cdot\bar{\nabla})\bar{v} = \bar{\nabla}(2\alpha\beta\rho) + \bar{\nabla}\left(\frac{4\beta^2\bar{\nabla}^2\sqrt{\rho}}{\sqrt{\rho}}\right)$$

This is the linear momentum continuity equation for a compressible, irrotational, inviscid (that is a superfluid) flow. The body forces at right-hand side depends only on the density (the fluid is barotropic) and the second term, which has no classical analogue is called quantum pressure.

Bogoliubov excitations

Let us look for infinitesimal wave-type excitations over the uniform solution in an unbounded domain ($V \equiv 0$). We will seek for solutions of

$$i\partial_t \psi + \beta \bar{\nabla}^2 \psi + \alpha |\psi|^2 \psi = 0$$

given $\psi(\bar{x},t) = \left[\sqrt{\rho_\infty} + \varepsilon \phi(\bar{x},t)\right] e^{-i\mu t}$, $\varepsilon \ll \sqrt{\rho_\infty}$

(meaning that we assume that at initial time $t=0$ the uniform solution is purely real $\psi(t=0) = \sqrt{\rho_\infty}$)

By plugging this ansatz into GP we get

$$\mu\sqrt{\rho_\infty}\, e^{-i\mu t} + \varepsilon i \partial_t \left(\phi e^{-i\mu t}\right) + \varepsilon \beta \bar{\nabla}^2 \left(\phi e^{-i\mu t}\right)$$

$$+ \alpha \rho_\infty \sqrt{\rho_\infty}\, e^{-i\mu t} + \varepsilon \alpha \rho_\infty \left(2\phi e^{-i\mu t} + \phi^* e^{-i\mu t}\right) + O(\varepsilon^2) = 0$$

At order ε^0:

$$\mu\sqrt{\rho_\infty}\,\cancel{e^{-i\mu t}} + \alpha \rho_\infty \sqrt{\rho_\infty}\,\cancel{e^{-i\mu t}} = \mu + \alpha \rho_\infty = 0$$

because of the definition $\mu = -\alpha \rho_\infty$

At order ε^1:

$$i\partial_t \phi\, \cancel{e^{-i\mu t}} + \mu \phi\, \cancel{e^{-i\mu t}} + \beta \bar{\nabla}^2 \phi\, \cancel{e^{-i\mu t}} + 2\alpha\rho_\infty \phi\, \cancel{e^{-i\mu t}} + \alpha \rho_\infty \phi^* \cancel{e^{-i\mu t}} = 0$$

$$i\partial_t \phi + \beta \bar{\nabla}^2 \phi + (\mu + 2\alpha \rho_\infty)\phi + (\alpha \rho_\infty)\phi^* = 0$$

$$i\partial_t \phi + \beta \bar{\nabla}^2 \phi + \alpha \rho_\infty \phi + \alpha \rho_\infty \phi^* = 0$$

and equivalently the equation for the c.c.

$$-i\partial_t \phi^* + \beta \bar\nabla^2 \phi^* + \alpha \rho_\infty \phi^* + \alpha \rho_\infty \phi = 0$$

We can then look for wave-type solutions in the form

$$\phi(\bar x, t) = A e^{i(\bar k\cdot \bar x - \omega t)} + B^* e^{-i(\bar k \cdot \bar x - \omega t)}, \quad A, B \in \mathbb{C}$$

When substituting into the two equations, and using for brevity the notation $e^{i(\bar k\cdot\bar x - \omega t)} = e^+$, $e^{-i(\bar k\cdot\bar x + \omega t)} = e^-$, one get

$$\omega A e^+ - \omega B^* e^- - \beta |\bar k|^2 A e^+ - \beta |\bar k|^2 B^* e^-$$
$$+ \alpha\rho_\infty A e^+ + \alpha\rho_\infty B^* e^- + \alpha\rho_\infty A^* e^- + \alpha\rho_\infty B e^+ = 0$$

$$e^+ \left(\omega A - \beta |\bar k|^2 A + \alpha\rho_\infty A + \alpha\rho_\infty B\right)$$
$$+ e^- \left(-\omega B^* - \beta |\bar k|^2 B^* + \alpha\rho_\infty B^* + \alpha\rho_\infty A^*\right) = 0$$

Because this must be true for any e^+ and e^- we need

$$\begin{cases} (\omega - \beta|\bar k|^2 + \alpha\rho_\infty) A + (\alpha\rho_\infty) B = 0 \\ (\alpha\rho_\infty) A^* + (-\omega - \beta|\bar k|^2 + \alpha\rho_\infty) B^* = 0 \end{cases}$$

from 2nd eq. $\Longrightarrow$ $A^* = -\dfrac{(-\omega - \beta|\bar k|^2 + \alpha\rho_\infty)}{(\alpha\rho_\infty)} B^* \Longrightarrow A = -\dfrac{(-\omega - \beta|\bar k|^2 + \alpha\rho_\infty)}{(\alpha\rho_\infty)} B$

which substituted in the first equation gives

$$\left(\omega-\beta|\bar{k}|^{2}+\alpha\rho_{\infty}\right)\left(-\omega-\beta|\bar{k}|^{2}+\alpha\rho_{\infty}\right)-\left(\alpha\rho_{\infty}\right)^{2}=0$$

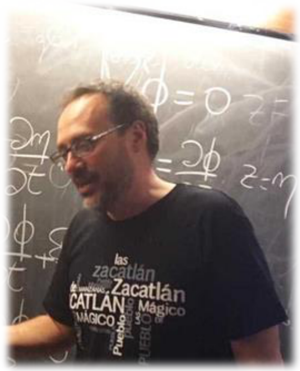
$$\left(-\beta|\bar{k}|^{2}+\alpha\rho_{\infty}\right)^{2}-\omega^{2}-\left(\alpha\rho_{\infty}\right)^{2}=0$$

$$\omega^{2}=\beta^{2}|\bar{k}|^{4}-2\alpha\beta\rho_{\infty}|\bar{k}|^{2}=\beta|\bar{k}|^{2}\left(\beta|\bar{k}|^{2}-2\alpha\rho_{\infty}\right)$$

$$\omega=\pm\sqrt{\beta}\,|\bar{k}|\sqrt{\beta|k|^{2}-2\alpha\rho_{\infty}}$$

GRAZIE PER L'ATTENZIONE!

Un grazie particolare ai miei collaboratori della scuola di Torino:



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Alberto Villois

Umberto Giuriato

Se siete interessati ad un progetto di tesi e/o ad un dottorato in questo ambito, contatteci!

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