Quasi-geostrophic approximation

Textbooks and web sites references for this lecture:

- James R. Holton, An Introduction to Dynamic Meteorology, Academic Press, 1992, ISBN 0-12-354355-X (§ 6.1-6.3)

The quasi-geostrophic approximation (Charney, 1947)

The q.g. approximation has been a fundamental simplification of the equations representing the extratropical large scale atmospheric circulation.

Pre-elaborated by Rossby and fully developed by Charney (1947, 1948) and Eady (1949), it allows an analytical treatment of the basic atmospheric motions in the case $Ro = U/fL \ll 1$.

Given $U \approx 10$ m/s, $f \approx 10^{-4} s^{-1}$, it means $L \gg U/f \Rightarrow L \approx 10^6$ m.

Physically, it means that the wind velocity obeys to the geostrophic relationship, but additional properties have to be satisfied (for example, the hydrostatic approximation).

The basic hypotheses are the requirements of hydrostatic and geostrophic balances.

The geostrophic approximation

- Neglecting friction, gravitation and inertial acceleration (=stationary flow: d**u**/dt=0) in the NS equations:

$$\cancel{\frac{d\vec{U}}{dt}} = -2\vec{\Omega}\times\vec{U} - \frac{1}{\rho}\overrightarrow{\nabla p} + \cancel{\vec{g}} + \cancel{\vec{F}}$$

- In cartesian coordinates (neglecting term with w):

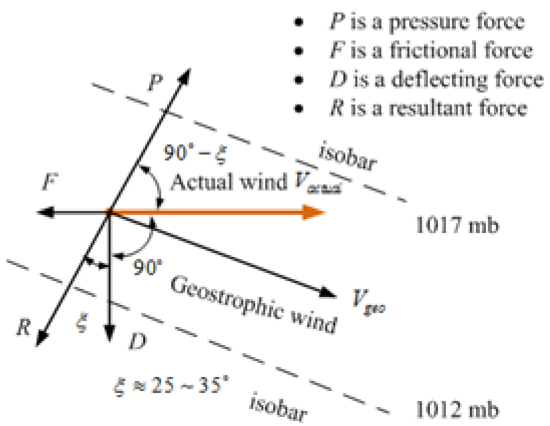
$$\left.\begin{aligned} \frac{1}{\rho}\frac{\partial p}{\partial x} &= 2\Omega v\cos\varphi \approx f_0 v & &\rightarrow v_g = \frac{1}{\rho f_0}\frac{\partial p}{\partial x} \\ \frac{1}{\rho}\frac{\partial p}{\partial y} &= -2\Omega u\sin\varphi \approx -f_0 u & &\rightarrow u_g = -\frac{1}{\rho f_0}\frac{\partial p}{\partial y} \end{aligned}\right\} \vec{u_g} = \vec{k}\times\frac{1}{\rho f_0}\overrightarrow{\nabla_H p}$$

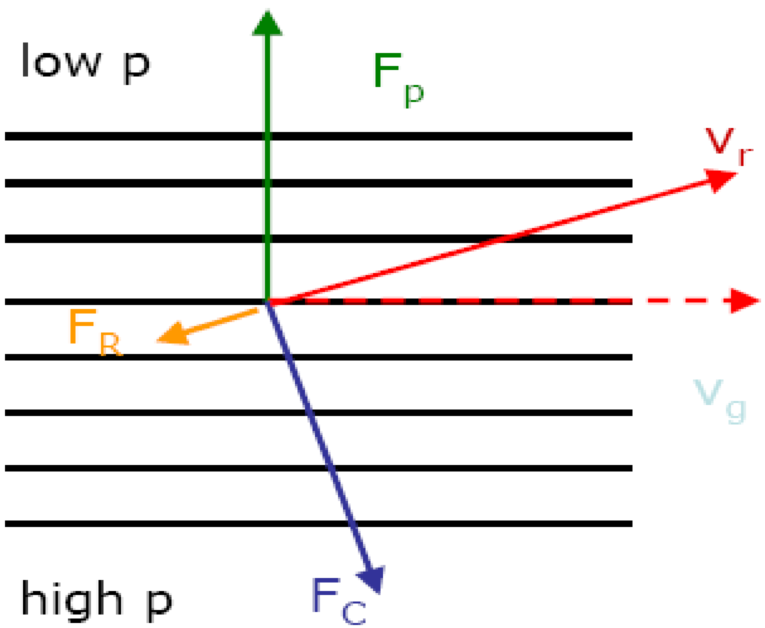
$$\left.\begin{aligned} v_g &= \frac{1}{\rho f_0}\frac{\partial p}{\partial x} = -\frac{1}{f_0}\frac{\partial \Phi}{\partial x} \\ u_g &= -\frac{1}{\rho f_0}\frac{\partial p}{\partial y} = \frac{1}{f_0}\frac{\partial \Phi}{\partial y} \end{aligned}\right\} \vec{u_g} = \vec{k}\times\frac{1}{f}\overrightarrow{\nabla\Phi}$$

where the approximation $f_0=2\Omega\sin\phi_0\approx 2\Omega\cos\phi_0$ is valid at middle latitudes ($\phi_0 \approx 45°$)

- This equation defines the geostrophic wind as the wind produced by the balance between Coriolis and (horizontal) pressure gradient acceleration
- The above equation are diagnostic, not prognostic

Geostrophic wind with friction





- Assume balance of pressure gradient, Coriolis and (turbulent) friction:

$$\cancel{\frac{d\vec{U}}{dt}} = -2\vec{\Omega}\times\vec{U} - \frac{1}{\rho}\overrightarrow{\nabla p} + \cancel{\vec{g}^*} + \vec{f}_r$$

- Angle of wind velocity depends on surface friction, e.g.:
  – ocean: 15°-30°
  – land: 25°-40°
  – “rough”land: 35°-50°

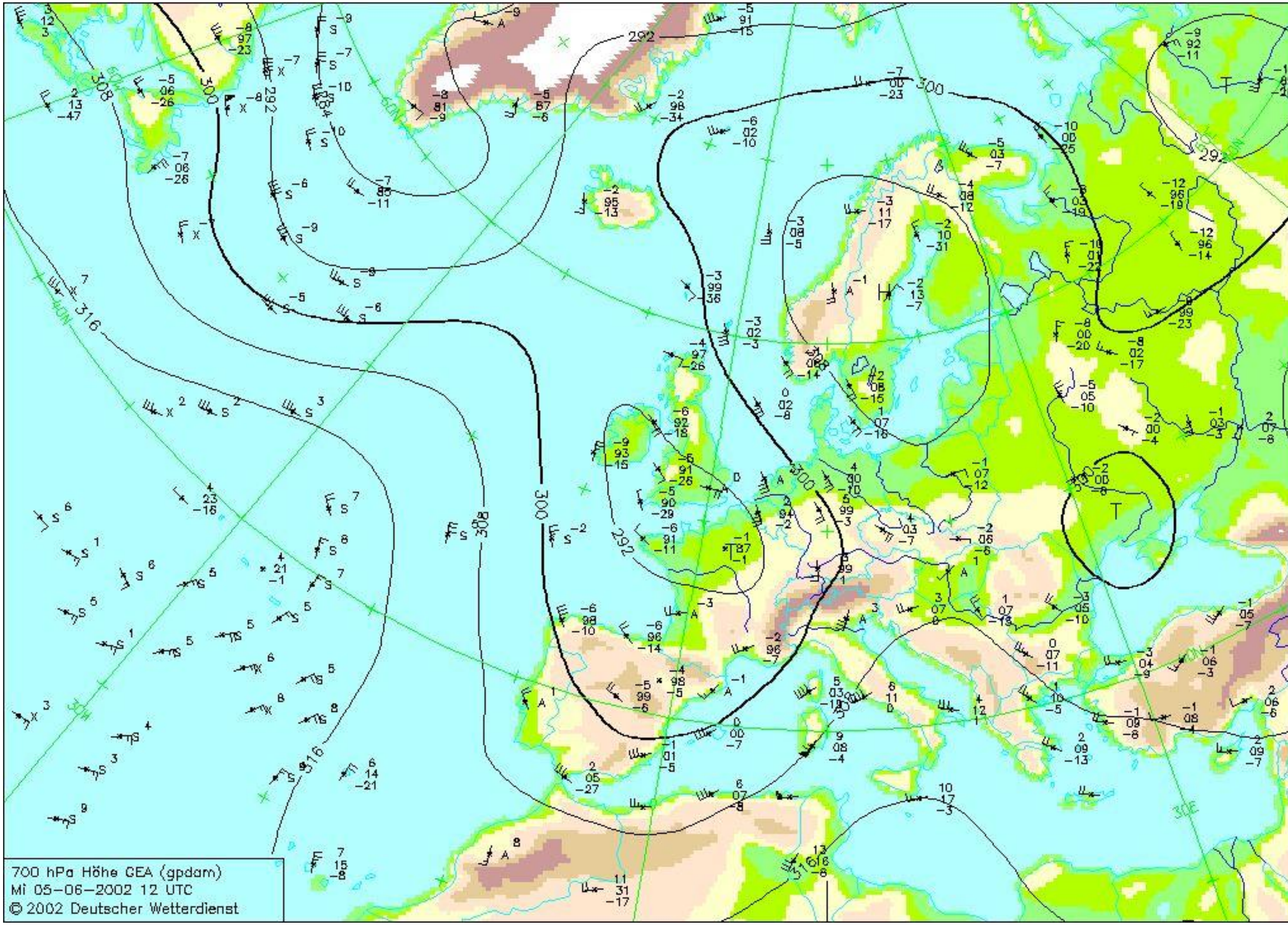
- Atmospheric circulation systems rarely resemble simple circular vortices and are generally highly asymmetric in form, with strongest winds and larger gradients concentrated along narrow bands called fronts, where there is strong baroclinicity

- Part of their complexity is due to the embeddment of the synoptic systems are embedded in a slowly varying planetary-scale flow highly baroclinic, part to the interaction with orography, and finally part to the dependance from the surface characteristics

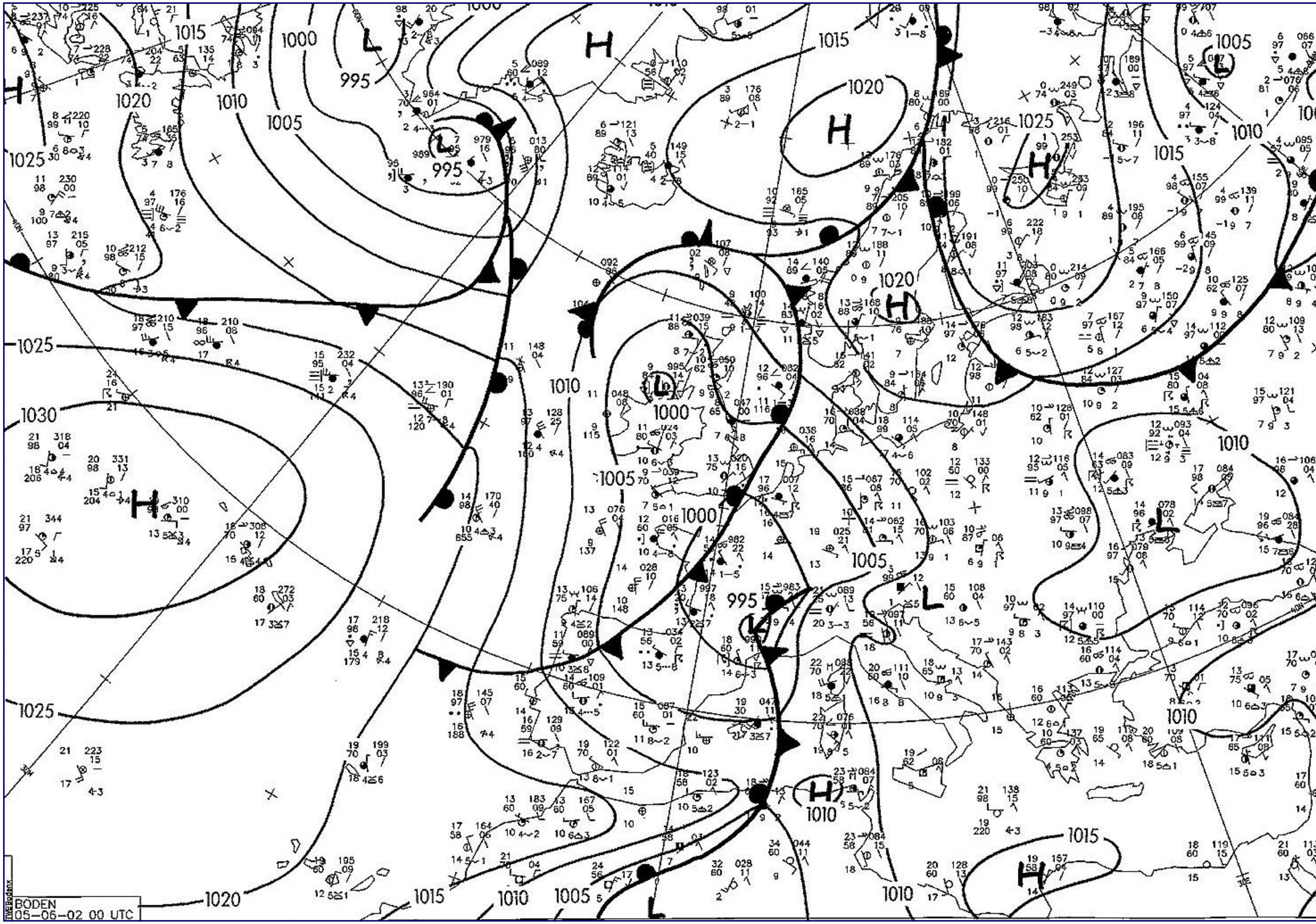
- Baroclinicity is source of instability as baroclinic disturbances may themselves act to intensify preexisting temperature gradients and hence generate frontal zones

- In this section, we want to investigate how observed structure of midlatitude synoptic systems can be interpreted in terms of constraints imposed on synoptic-scale motions by dynamical equations. Specifically, we want to see that hydrostatic and geostrophic flows are completely understood by knowing geopotential distribution

The basic notion of geostrophic equilibrium can be easily verified in the **free** atmosphere...:



but not so well at the **surface**:



The q.g. approximation can be formally obtained by expanding the equations of motion in terms of the Rossby number, and retaining the first order terms.

A generic variable X is written:

$$X = X^{(0)} + Ro\, X^{(1)} + Ro^2\, X^{(2)} + \dots$$

However, additional assumptions have to be made on the atmospheric stratification properties.

To do that, it is convenient to express the thermodynamic variables p, θ, ω as a horizontal average component (function only of height) $p_0(z)$, $\theta_0(z)$, $\omega_0(z)$ and their deviations.

The basic components represent a hydrostatic, stably stratified atmosphere.

The primitive equations

Let's consider the **primitive equations** of motion in isobaric coordinates and simplified conditions: inviscid, adiabatic, hydrostatic (the latter certainly valid for L>10 km):

$$\frac{d\vec{v}}{dt} + f\vec{k} \times \vec{v} = -\nabla\Phi \qquad \nabla \circ \vec{v} + \frac{\partial \omega}{\partial p} = 0$$

$$\frac{\partial \Phi}{\partial p} = -\alpha = -\frac{RT}{p} \qquad \left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla\right) T - S_p \omega = \frac{J}{c_p}$$

Where:

$$\frac{d}{dt} = \left(\frac{\partial}{\partial t}\right)_p + \left(\vec{v} \cdot \nabla\right)_p + \omega \frac{\partial}{\partial p} \qquad \omega = \frac{dp}{dt} \qquad S_p = -T \frac{\partial \ln \theta}{\partial p}$$

Separate wind in geostrophic and ageostrophic component: $\vec{v} = \overrightarrow{v_g} + \overrightarrow{v_a}$

As meridional length scale L is small if compared with Earth radius, it is possible to define geostrophic wind by using constant reference value f_0 for Coriolis parameter:

$$\overrightarrow{v_g} = \frac{\vec{k} \times \nabla \Phi}{f_0}$$

For the systems of our interest $\left|\overrightarrow{v_g}\right| >> \left|\overrightarrow{v_a}\right|$ i.e. $\left|\overrightarrow{v_a}\right| / \left|\overrightarrow{v_g}\right| \approx O(R_o)$, that means that in the momentum equation **v** can be replaced by $\mathbf{v}_g$ and that the vertical advection can be neglected:

then d**v**/dt~$d_g\mathbf{v}_g$/dt where: $\dfrac{d_g}{dt} = \dfrac{\partial}{\partial t} + \overrightarrow{v_g} \cdot \nabla$

Although a constant f_0 has been used in defining $\mathbf{v}_g$, we need to keep variation of f in the Coriolis term, then we use the "**Beta-plane approximation**": we pose $f=f_0+\beta(y-y_0)$, where

$\beta = (df/dy)_0 = 2\Omega\cos\phi_0/R$ ($dy=Rd\phi$). [$\beta \approx 10^{-11}$ m^{-1} s^{-1}]

Note that

$$\frac{\beta L}{f_0} \approx \frac{\cos\phi_0}{\sin\phi_0}\frac{L}{a} \approx O(R_o)$$

The right part of momentum equation can be written as:

$$f\vec{k}\times\vec{v}+\nabla\Phi=(f_0+\beta y)\vec{k}\times\left(\overrightarrow{v_g}+\overrightarrow{v_a}\right)-f_0\vec{k}\times\overrightarrow{v_g}\approx f_0\vec{k}\times\overrightarrow{v_a}+\beta y\vec{k}\times\overrightarrow{v_g}$$

And the momentum equation becomes:

$$\frac{d_g\overrightarrow{v_g}}{dt}=-f_0\vec{k}\times\overrightarrow{v_a}-\beta y\vec{k}\times\overrightarrow{v_g}$$

all term considered are $O(R_o)$
all term neglected are $O(R_o^2)$ or smaller

As geostrophic wind is non-divergent $\nabla\cdot\vec{v}=\nabla\cdot\overrightarrow{v_a}=\frac{\partial u_a}{\partial x}+\frac{\partial v_a}{\partial y}$ → $\nabla\cdot\overrightarrow{v_a}+\frac{\partial\omega}{\partial p}=0$

This continuity equation shows that ω is determined only by ageostrophic wind.
Now, let consider temperature field T_{tot} as composed by a basic state T_0 depending only on pressure p and a deviation T(x,y,z,t):

$$T_{tot} = T_0(p) + T(x,y,z,t)$$

as $|dT_0/dp|>>|\partial T/\partial p|$ we can rewrite thermodynamic equation as:

$$\left(\frac{\partial}{\partial t}+\overrightarrow{v_g}\cdot\nabla\right)T-\left(\frac{\sigma p}{R}\right)\omega=\frac{J}{c_p},\qquad \sigma\equiv-\frac{RT_0}{p}\frac{d\ln\theta_0}{dp}\approx 2\ 10^{-6}\quad m^2Pa^{-2}s^{-2}$$

Quasi-geostrophic equations

The five equations:

$$\overrightarrow{v_g} = \frac{\vec{k} \times \nabla\Phi}{f_0}$$

$$\nabla \cdot \overrightarrow{v_a} + \frac{\partial \omega}{\partial p} = 0$$

$$\left(\frac{\partial}{\partial t} + \overrightarrow{v_g} \cdot \nabla \right) T - \left(\frac{\sigma p}{R} \right) \omega = \frac{J}{c_p}$$

$$\frac{\partial \Phi}{\partial p} = -\alpha = -\frac{RT}{p}$$

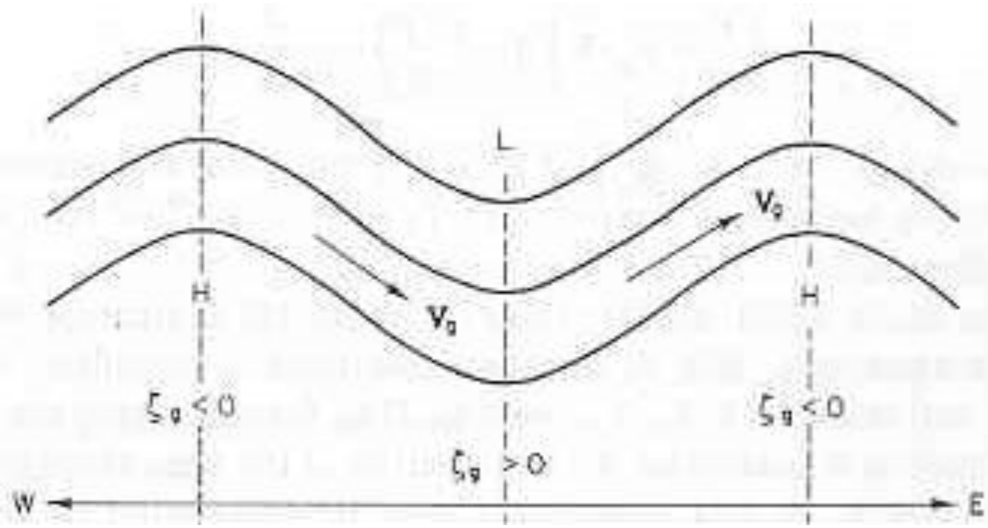
$$\frac{d_g \overrightarrow{v_g}}{dt} = -f_0 \vec{k} \times \overrightarrow{v_a} - \beta y \vec{k} \times \overrightarrow{v_g}$$

constitute the quasi-geostrophic equations.
Provided J known, this set of equations is complete.

Quasi-geostrophic vorticity equation

Defining the geostrophic vorticity $\zeta_g = \mathbf{k} \circ \nabla \times \mathbf{v}_g$, it is possible to express as:

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f_0}\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right) = \frac{\nabla^2 \Phi}{f_0}$$



This equation can be used to determine geostrophic vorticity is geopotential field is known, or vice-versa. This second case is one reason for which this equation is so famous

As Laplacian of a function is maximum where function is minimum, positive vorticity implies low values of geopotential ,and vice versa

X and Y components of momentum equation can be written as:

$$\frac{d_g u_g}{dt} - f_0 v_a - \beta y v_g = 0 \qquad \frac{d_g v_g}{dt} + f_0 u_a + \beta y u_g = 0$$

Deriving the two above equations for y and x respectively, we arrive to:

$$\frac{d_g \zeta_g}{dt} = \frac{d_g}{dt}\left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right) = \frac{\partial}{\partial x}\left(-f_0 u_a - \beta y u_g\right) - \frac{\partial}{\partial y}\left(f_0 v_a + \beta y v_g\right) =$$

$$= -f_0\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right) - \beta y\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right) - \beta v_g$$

Now, splitting the total time derivative of ζ_g : $\frac{d\zeta_g}{dt} = \frac{\partial \zeta_g}{\partial t} + \overrightarrow{v_g} \cdot \nabla \zeta_g$

Regarding planetary vorticity: $\frac{d_g f}{dt} = \overrightarrow{v_g} \cdot \nabla f = \beta v_g$

Thus it is possible to combine vorticities: $\frac{d\zeta_g}{dt} = \frac{d\left(\zeta_g + f\right)}{dt} = \frac{\partial \zeta_g}{\partial t} + \overrightarrow{v_g} \cdot \nabla \zeta_g + \beta v_g$

Since vorticity advection can be rewritten as:

$$\overrightarrow{v_g} \cdot \nabla\left(\zeta_g + f\right) = \overrightarrow{v_g} \cdot \nabla \zeta_g + \beta v_g$$

and since, from continuity equation:

$$\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right) = -\frac{\partial \omega}{\partial p}$$

then the geostrophic vorticity equation can be rewritten as:

$$\frac{\partial \zeta_g}{\partial t} = -\overrightarrow{v_g} \cdot \nabla \zeta_g - \beta v_g + f_0 \frac{\partial \omega}{\partial p}$$

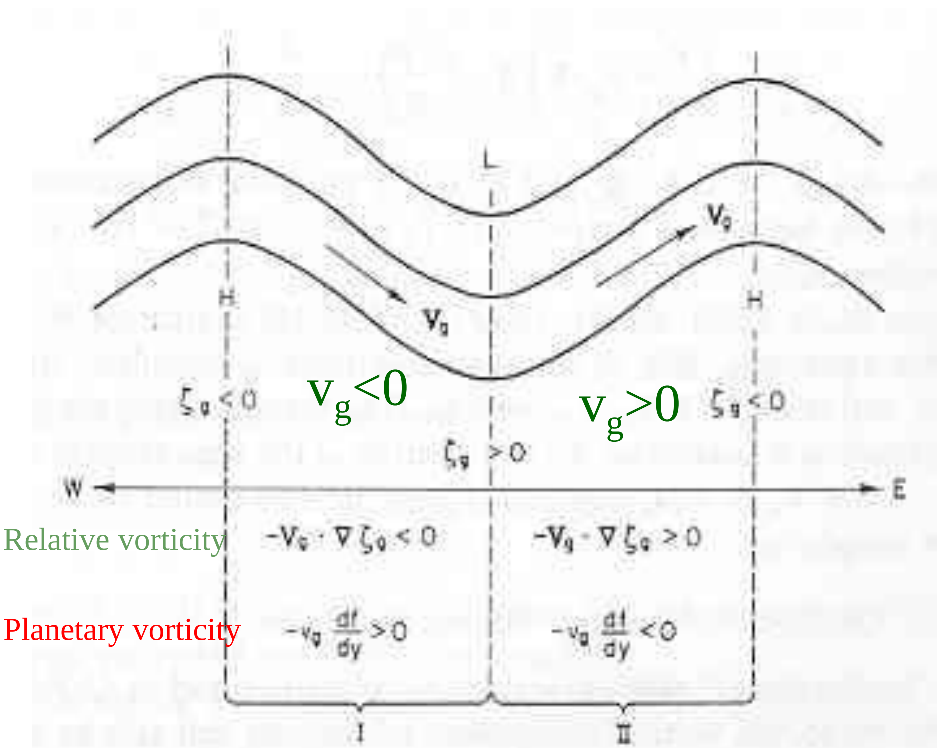
Rate of change of geostrophic vorticity

Geostrophic advection of relative vorticity

Advection of absolute vorticity by geostrophic wind

Geostrophic advection of planetary vorticity

Concentration or dilution of vorticity by stretching or shrinking of fluid columns (divergence effect) geostrophic wind



In region I, upstream of 500 hPa trough, the geostrophic wind is directed from negative vorticity maximum at the ridge toward the positive vorticity maximum at the trough, so that advection of relative vorticity is negative (bring anticyclonic curvature).

At the same time, being $v_g<0$ in region I, this geostrophic component is downgradient of planetary vorticity, so advection of planetary vorticity is positive. In this region two terms have contrary effects. Vice versa in region II.

Advection of relative vorticity tends to move the vorticity pattern (i.e. troughs and rigdes) **eastwards** (downstream). But **advection of planetary vorticity tends to move troughs and rigdes westwards**, against the advecting wind field (*retrograde motion or retrogression*).

The net effect of advection on the evolution of the vorticity pattern depends upon which type of vorticity advection dominates.

In order to compare the magnitudes of relative and planetary vorticity advections, we split geopotential field Φ in a time and zonally averaged part <Φ(x,p)> and a fluctuating part that has a sinusoidal dependance in x and y:

$$\Phi(x, y, p, t) = \overline{\Phi}(y, p) + \Phi'(p, t)\sin kx \cos ly, \qquad \overline{\Phi}(y, p) = \Phi_0(p) - f_o U y$$

Wave numbers k and l are defined as:

$k=2\pi/L_x$ and $l=2\pi/L_y$

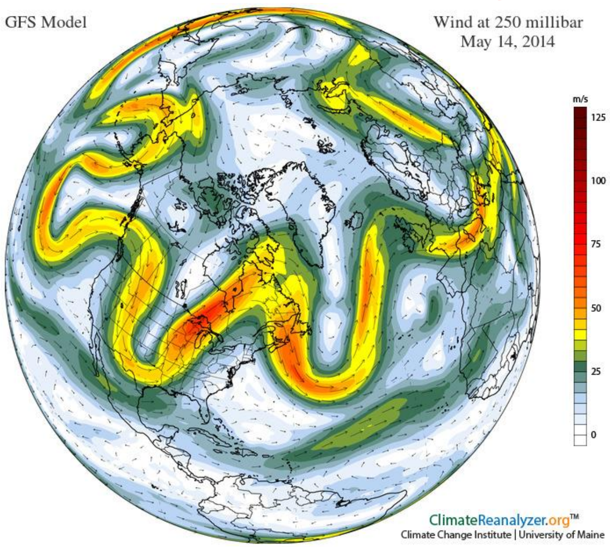
where L_x and L_y are wavelenghts in x and y directions. Φ_0 is the standard atmosphere distribution, U the mean constant zonal wind. The geostrophic vorticity is then:

$$\zeta_g = \frac{\nabla^2\Phi}{f_0} = -\frac{\left(k^2 + l^2\right)}{f_0}\Phi' \sin kx \cos ly = -\frac{\left(k^2 + l^2\right)}{f_0}\left(\Phi - \overline{\Phi}\right)$$

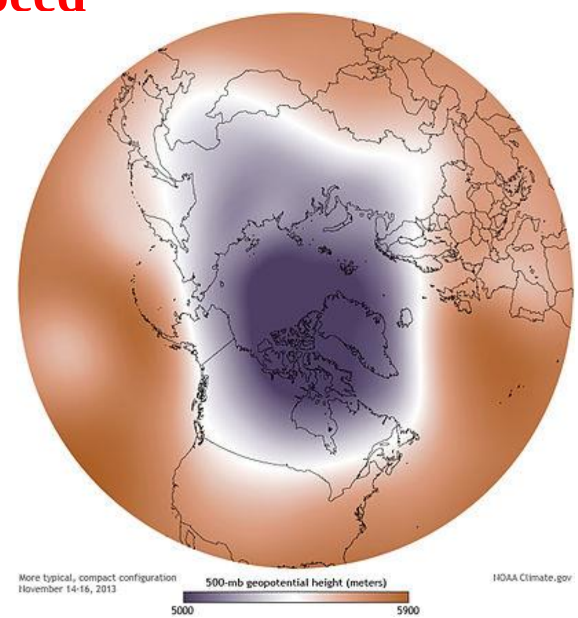
For a disturbance with a given amplitude of geopotential disturbance Φ', the amplitude of the vorticity increases as square of the wave number, or inversely to square of the horizontal scale. As a consequence, the advection of relative vorticity dominates over planetary vorticity advection for short waves ($L_x < 3000$ km), while for long waves ($L_x > 10000$ km) the planetary vorticity advection tends to dominate.

Therefore, as general rule, **short-wavelength synoptic-scale systems should move eastwards with the advecting zonal flow, while long planetary waves should tend to retrogress** (move westwards against mean flow)

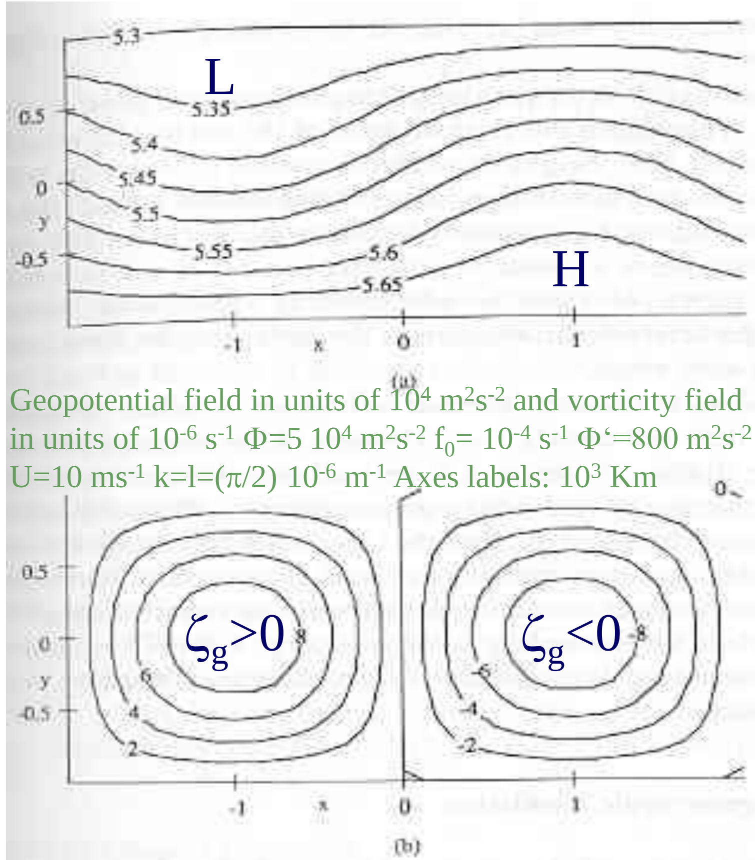
Waves of intermediate wavelength may be **quasi-stationary or move eastwards much slower than the mean geostrophic wind speed**



Short-wavelength system



Long-wavelength system



Geopotential field in units of 10^4 m^{2}s^{-2} and vorticity field in units of 10^{-6} s^{-1} Φ=5 10^4 m^{2}s^{-2} f_0= 10^{-4} s^{-1} Φ'=800 m^{2}s^{-2} U=10 ms^{-1} k=l=(π/2) 10^{-6} m^{-1} Axes labels: 10^3 Km

Since **positive maxima in relative vorticity are associated with cyclonic disturbances**, regions of positive vorticity advection, which can easily be estimated from upper-level maps, are commonly used as aids in forecasting synoptic-scale weather disturbances

- Vorticity advection does not alone determine evolution of meteorological systems

- A **change in the vertical shear of horizontal wind speed** associated with differential (i.e. height-dependent) vorticity advection will **drive an ageostrophic vertical circulation, which adiabatically adjusts the horizontal temperature gradient** in order to maintain thermal wind balance

- The **convergence and divergence fields associated with vertical circulation** will not only **modify the effects of vorticity advection at upper levels**, but also **force changes in the vorticity distribution in the lower troposphere**, where advection may be very weak

- In analogous manner, **thermal advection**, which is often **strong near surface**, does not merely force changes in temperature in the lower troposphere, but **induces a vertical circulation which through its associated divergence and convergence patterns alters the vorticity fields both near the surface and aloft**, so that thermal wind balance is maintained

- The **vertical circulation induced by quasi-geostrophic differential vorticity advection and thermal advection is generally an order of magnitude larger than that induced by boundary-layer pumping**. Thus, it is reasonable to neglect boundary layer effects to a first approximation in quasi-geostrophic theory