Lecture 11: Dynamics

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The Navier-Stokes equations

- The above equations are, in truth, three and are known as Navier-Stokes Equations (NSEs)
- The NSEs (or Euler equations) are non-linear because they are *quadratic* in U:

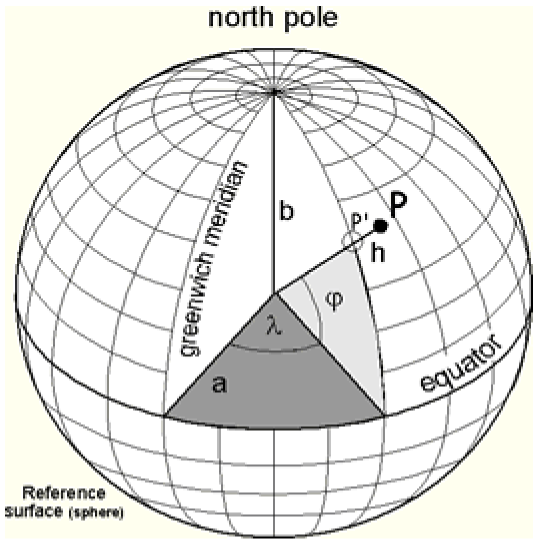
$$\frac{d\vec{U}}{dt} = \frac{\partial \vec{U}}{\partial t} + \left(\frac{\partial \vec{r}}{\partial t} \nabla \right) \vec{U} = \frac{\partial \vec{U}}{\partial t} + \left(\vec{U} \nabla \right) \vec{U} = \frac{\partial \vec{U}}{\partial t} + \frac{1}{2} \overrightarrow{\nabla U^2} - \left[\vec{U} \times \left(\vec{\nabla} \times \vec{U} \right) \right]$$

- This non-linearity causes *chaotic phenomena* in the atmosphere

NSEs in polar coordinates

- **Earth is spherical, thus it is convenient to develop a method able to represent the equation of motions in coordinates adequate to this geometry, i.e. polar spherical coordinates**
- **The polar spherical coordinates on Earth are:**
  - Longitude λ, versor *i* positive in W→E direction
  - Latitude φ, versor *j* positive in S→N direction
  - Quote z = r – a (heigth on the surface of the tangent local plan), versor k positive upwards
- **The relative velocity u is expressed as:**

$$\boldsymbol{u} = \boldsymbol{i}\, u + \boldsymbol{j}\, v + \boldsymbol{k}\, w$$



NSEs in polar coordinates

- The three components u, v and w are defined by:

$$u = r\cos\varphi \frac{d\lambda}{dt}, \quad v = r\frac{d\varphi}{dt}, \quad w = \frac{dz}{dt}$$

r being the distance of point P from the center of the Earth, and a the radius of the Earth, with r=a+z

- Being z<<a, generally it is r = a = const
- The above equations imply:

$$dx = r\cos\varphi\, d\lambda, \quad dy = r\, d\varphi$$

Velocity derivative on rotating system

- The velocity derivative of the wind vector **u** on a rotating system is composed not only by the derivative of its components (u, v and w, given by the NSEs), but also by the time derivative of the reference system versors:

$$\frac{d\vec{u}}{dt} = \vec{i}\,\frac{du}{dt} + \vec{j}\,\frac{dv}{dt} + \vec{k}\,\frac{dw}{dt} + u\frac{d\vec{i}}{dt} + v\frac{d\vec{j}}{dt} + w\frac{d\vec{k}}{dt}$$

- Let's analyze the time derivative of each versor, remembering that

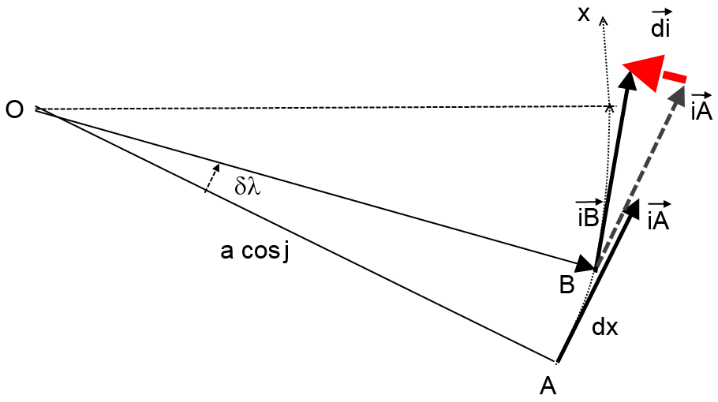
$$\frac{d}{dt} = \frac{\partial}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)$$

Versor i

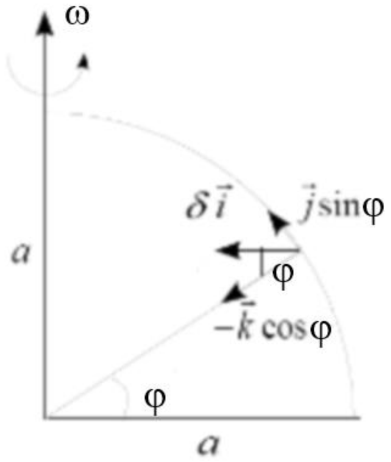
$$\frac{d\vec{i}}{dt} = \cancel{\frac{\partial \vec{i}}{\partial t}} + u\frac{\partial \vec{i}}{\partial x} + v\cancel{\frac{\partial \vec{i}}{\partial y}} + w\cancel{\frac{\partial \vec{i}}{\partial z}}$$

- **Regarding i, being |∂i|=δλ and |∂x|=a cosφ δλ , it is:**

$$\left|\frac{d\vec{i}}{dt}\right| = u\left|\frac{\partial \vec{i}}{\partial x}\right| = \frac{u}{a\cos\varphi}$$

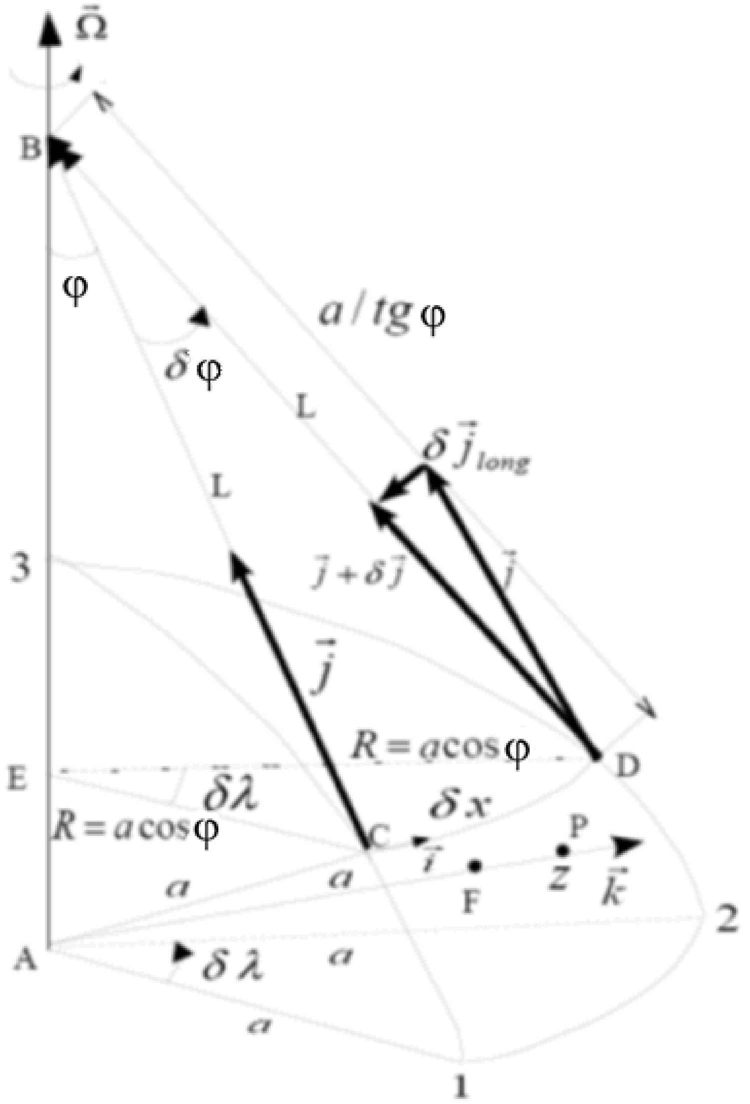


- **Its components are:**



$$\frac{d\vec{i}}{dt} = \frac{u}{a\cos\varphi}\left(\vec{j}\sin\varphi - \vec{k}\cos\varphi\right)$$

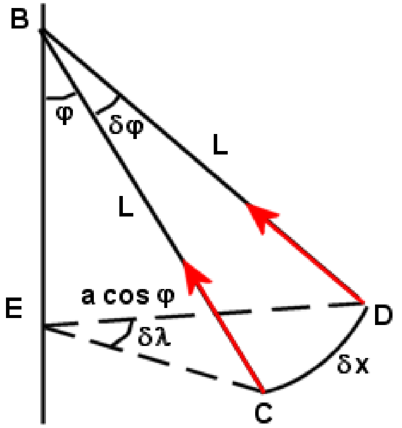
Versor j (1/2)



- **j is function only of x and y, thus ∂j is composed by a $\partial \mathbf{j}_{lo}$ (along x) and a $\partial \mathbf{j}_{la}$ (y)**

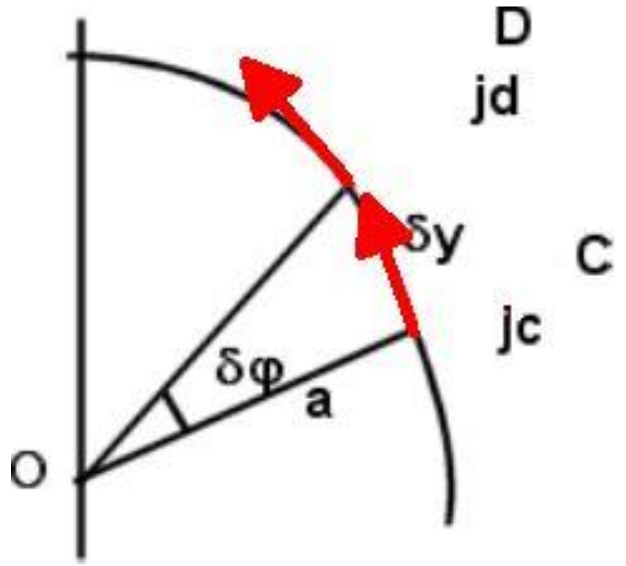
$$\frac{d\vec{j}}{dt}=\cancel{\frac{\partial\vec{j}}{\partial t}}+u\frac{\partial\vec{j_{lo}}}{\partial x}+v\frac{\partial\vec{j_{la}}}{\partial y}+\cancel{w\frac{\partial\vec{j}}{\partial z}}$$

- **Looking at «triangle» BCD: δx = L δφ**
- **But also EC = a cosφ = L sinφ thus L=a / tgφ then δx = L δφ = a δφ/tgφ**
- **Since $\delta \mathbf{j}_{lo}$ = δφ and $\delta \mathbf{j}_{lo}$ is antiparallel to i:**



$$\frac{\partial_{lo}\vec{j}}{\partial x}=-\frac{tg\varphi}{a}\vec{i}$$

Versor j (2/2)



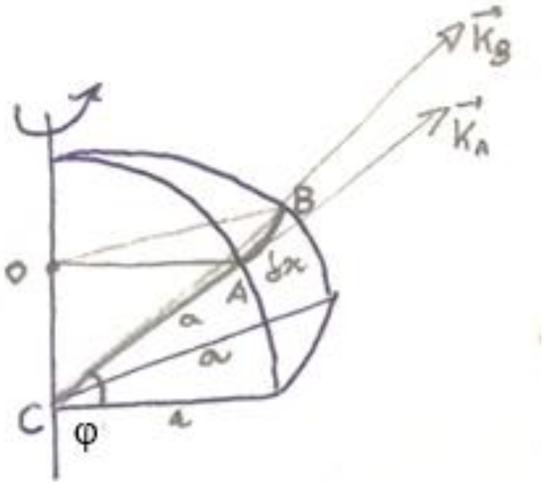
- **Moving northwards (at constant longitude, varying just the latitude, it is:**
  - $|\delta \mathbf{j}_{la}|=|\delta\varphi|$
  - $\delta y=a\delta\varphi$
- **Thus:**

$$\frac{\partial_{la}\vec{j}}{\partial y}=-\frac{\vec{k}}{a}$$

- **and the complete derivative is:**

$$\frac{d\vec{j}}{dt}=u\frac{\partial_{lo}\vec{j}}{\partial x}+v\frac{\partial_{la}\vec{j}}{\partial y}=-u\frac{tg\varphi}{a}\vec{i}-\frac{v}{a}\vec{k}$$

Versor k (1/2)

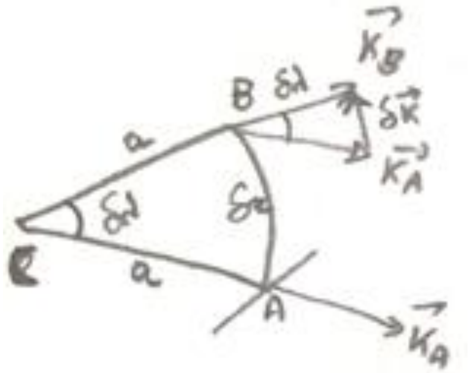


- Similarly to versor **j: k** is function only of x and y
- Thus ∂**k** is composed by two components: $\partial \mathbf{k}_{lo}$ (along x) and $\partial \mathbf{k}_{la}$ (along y)

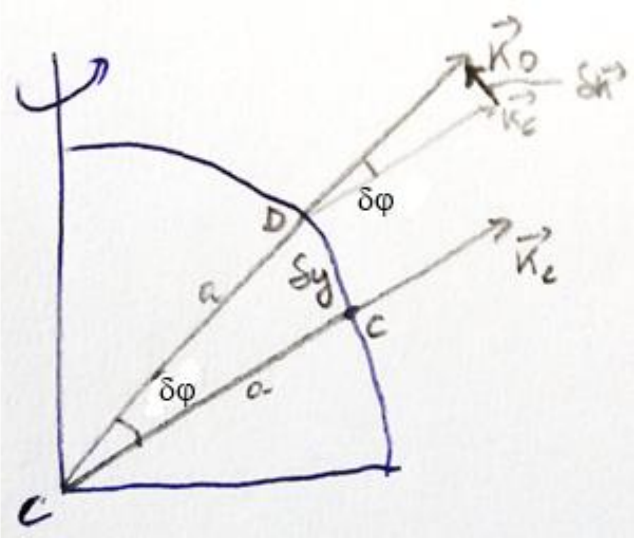
$$\frac{d\vec{j}}{dt} = \cancel{\frac{\partial \vec{j}}{\partial t}} + u\frac{\partial \overrightarrow{j_{lo}}}{\partial x} + v\frac{\partial \overrightarrow{j_{la}}}{\partial y} + \cancel{w\frac{\partial \vec{j}}{\partial z}}$$

- Moving A→B (along x, at constant longitude), considering the «triangle» OAB, we see that: |δ**k**| ~ $\delta\lambda$ and δx ~ a $\delta\lambda$.
- Thus:

$$\frac{\partial_{lo}\vec{k}}{\partial x} = \frac{1}{a}\vec{i}$$



Versor k (2/2)



- Now, moving C→D (along y, at constant lat itude), considering the «triangle» OCD:
  - $|\delta \mathbf{k}_{la}| \sim \delta\varphi$
  - $\delta y = a\, \delta\varphi$
- Thus the derivative is:

$$\frac{\partial_{la}\vec{k}}{\partial y} = \frac{1}{a}\vec{j}$$

- Finally, the complete derivative is:

$$\frac{d\vec{k}}{dt} = u\frac{\partial_{lo}\vec{k}}{\partial x} + v\frac{\partial_{la}\vec{k}}{\partial y} = \frac{u}{a}\vec{i} + \frac{v}{a}\vec{j}$$

Total derivative of u

- The three derivatives can thus be incorporated in the total derivative of **u** (slide 5):

$$\frac{d\vec{i}}{dt}=\frac{u}{a\cos\varphi}\left(\vec{j}\sin\varphi-\vec{k}\cos\varphi\right)$$

$$\frac{d\vec{j}}{dt}=-u\frac{tg\varphi}{a}\vec{i}-\frac{v}{a}\vec{k}$$

$$\frac{d\vec{k}}{dt}=\frac{u}{a}\vec{i}+\frac{v}{a}\vec{j}$$

$$\frac{d\vec{u}}{dt}=\left(\frac{du}{dt}-\frac{uvtg\theta}{a}+\frac{uw}{a}\right)\vec{i}+\left(\frac{dv}{dt}+\frac{u^2tg\theta}{a}+\frac{wv}{a}\right)\vec{j}+\left(\frac{dw}{dt}-\frac{u^2+v^2}{a}\right)\vec{k}$$

Forces in polar coordinates

- Remembering:

$$dx = a\cos\varphi\, d\lambda, \quad dy = a\, d\varphi$$

- Pressure gradient force

$$-\frac{\overrightarrow{\nabla p}}{\rho} = -\frac{1}{\rho}\frac{\partial p}{\partial x}\vec{i} - \frac{1}{\rho}\frac{\partial p}{\partial y}\vec{j} - \frac{1}{\rho}\frac{\partial p}{\partial z}\vec{k} = -\frac{1}{\rho a\cos\varphi}\frac{\partial p}{\partial \lambda}\vec{i} - \frac{1}{\rho a}\frac{\partial p}{\partial \varphi}\vec{j} - \frac{1}{\rho}\frac{\partial p}{\partial z}\vec{k}$$

- Apparent gravitation force:

$$\vec{g} = -\frac{\partial \Phi}{\partial x}\vec{i} - \frac{\partial \Phi}{\partial y}\vec{j} - \frac{\partial \Phi}{\partial z}\vec{k} = -\frac{1}{a\cos\varphi}\frac{\partial \Phi}{\partial \lambda}\vec{i} - \frac{1}{a}\frac{\partial \Phi}{\partial \varphi}\vec{j} - \frac{\partial \Phi}{\partial z}\vec{k}$$

(where the first two terms can be neglected)

The NSEs splitted in 3 components

- The complete equations can be so summarized (gravitation horizontal terms have been neglected):

$$\frac{du}{dt}-\frac{uv\,tg\varphi}{a}+\frac{uw}{a}=-\frac{1}{\rho a\cos\varphi}\frac{\partial p}{\partial\lambda}+2\left(\Omega v\sin\varphi-\Omega w\cos\varphi\right)+f_x$$

$$\frac{dv}{dt}+\frac{u^2tg\varphi}{a}+\frac{wv}{a}=-\frac{1}{\rho a}\frac{\partial p}{\partial\varphi}-2\Omega u\sin\varphi+f_y$$

$$\frac{dw}{dt}-\frac{u^2+v^2}{a}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-\frac{\partial\Phi}{\partial z}+2\Omega u\cos\varphi+f_z$$

(I) (II) (III) (IV) (V) (VI) (VII)

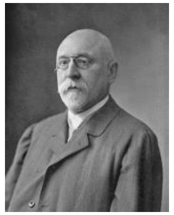
The NSEs splitted in 3 components

- **The meaning of the terms are:**
  - (I) inertial acceleration
  - (II) apparent acceleration due to curvature
  - (III) apparent acceleration due to curvature
  - (IV) pressure gradient acceleration
  - (V) gravitational acceleration
  - (VI) Coriolis acceleration
  - (VII) friction acceleration
- **Terms II and III represent the geometric correction to the relative acceleration in the non-inertial terrestrial system**
  - They do not depend on the Earth rotation
  - They depend only on the curvature of the coordinate system → for this reason, they can be neglected for synoptic motions at middle latitudes

Classification of the NSEs

- **The behavior of solutions of the NSEs can be classified using dimensionless numbers:**

- **Strouhal Number**



Vincenc Strouhal (1850-1922) Czech environmental physicist

$$St = \frac{\text{Inertial force}}{\text{Force of movement}} = \frac{\left|\left(\vec{U}\cdot\vec{\nabla}\right)\vec{U}\right|}{\left|\frac{\partial\vec{U}}{\partial t}\right|}$$

- **Rossby Number**

Carl-Gustaf Arvid Rossby (1898–1957) Swedish-born American meteorologist

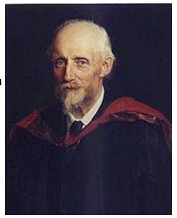
$$Ro = \frac{\text{Inertial force}}{\text{Coriolis force}} = \frac{\left|\left(\vec{U}\cdot\vec{\nabla}\right)\vec{U}\right|}{\left|2\left(\vec{U}\times\vec{\Omega}\right)\right|}$$

- **Euler Number**

Leonhard Euler (1707–1783) Swiss mathematician, physicist, astronomer, geographer, logician and engineer

$$Eu = \frac{\text{Inertial force}}{\text{Pressure gradient force}} = \frac{\left|\left(\vec{U}\cdot\vec{\nabla}\right)\vec{U}\right|}{\left|\frac{1}{\rho}\overrightarrow{\nabla p}\right|}$$

- **Reynolds Number**



Osborne Reynolds FRS (1842–1912) physicist and fluid dynamics scientist

$$\mathrm{Re} = \frac{\text{Inertial force}}{\text{Frictional force}} = \frac{\left|\left(\vec{U}\cdot\vec{\nabla}\right)\vec{U}\right|}{\left|\left(K+\nu\right)\cdot\overrightarrow{\Delta U}\right|}$$

Reynolds-Number and turbulence

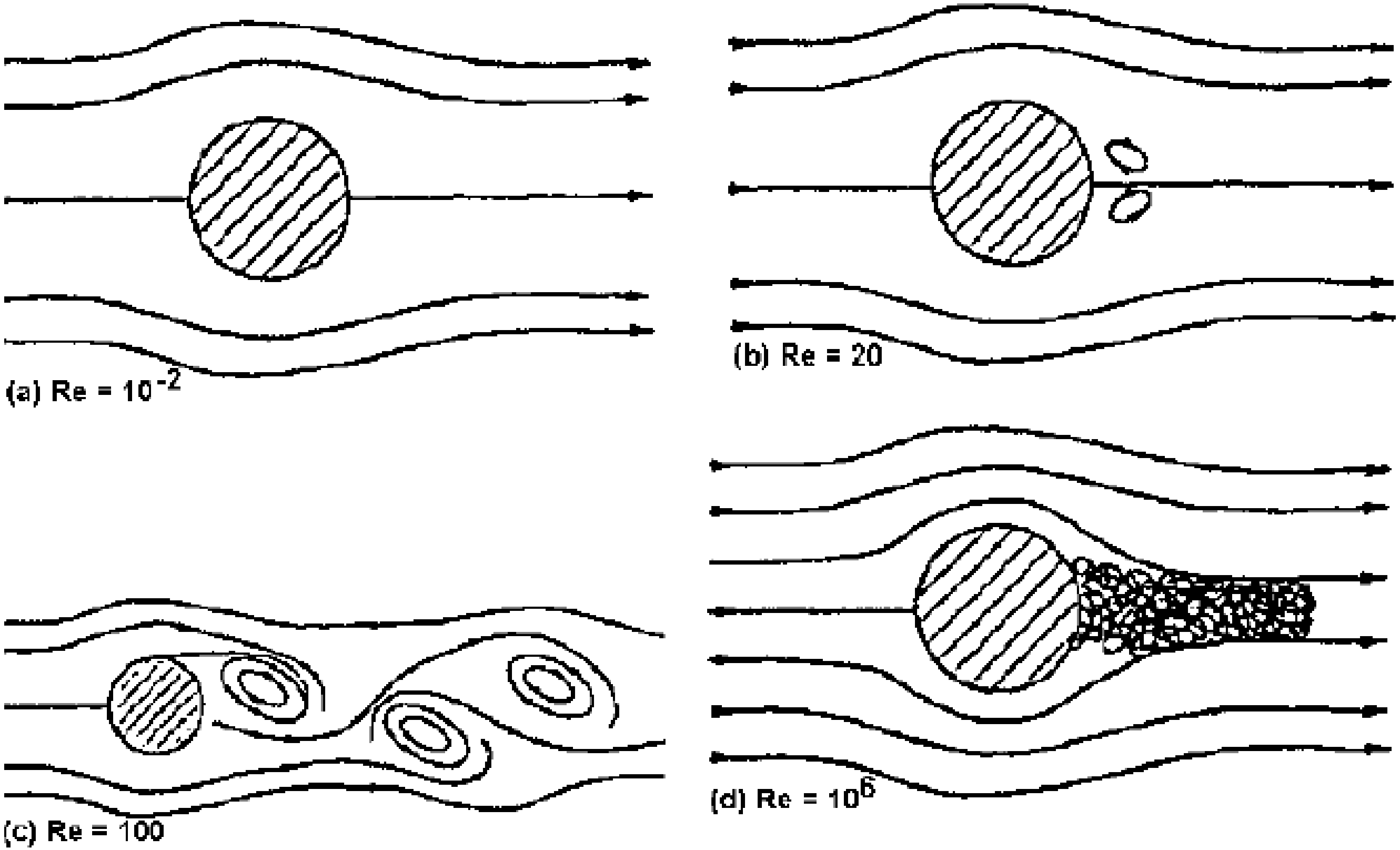
- The Reynolds-Number

$$\mathrm{Re} = \frac{\left|\left(\vec{U}\cdot\vec{\nabla}\right)\vec{U}\right|}{\left|\left(K+\nu\right)\cdot\overrightarrow{\Delta U}\right|} \approx \frac{|U|^2 / l}{\left(K+\nu\right)|U| / l^2} = \frac{|U|\,l}{K+\nu}$$

(l being the length scale of the system) provides an estimate for laminar or turbulent behavior of a fluid

- Critical Reynolds numbers:
  - $\mathrm{Re}_{crit} \approx 50{,}000$ for flat surfaces
  - $\mathrm{Re}_{crit} \approx 2{,}300$ for tubular structures

Reynolds-Number and turbulence



Reynolds-Number and turbulence

- **Typical Reynolds numbers in the atmosphere:**
  - Viscosity: ν= 0.15 cm^2/s
  - Typical velocity: U = 5 m/s
  - Typical length scale: l = 1000 m
- **→ Re ≈ 3 10^8 → quite high → atmosphere is governed by turbulence**
- **Length scale above which turbulence occurs is**

$$l_{crit} = \frac{\nu \,\mathrm{Re}_{crit}}{U} \approx 3\,\mathrm{mm}$$

- **→ Molecular-viscous flow and dissipation of energy is important only at length scales below ≈ 3 mm**

Synoptic scale analysis

- Scaling is a technique useful to determine whether some terms in the equation are negligible for motions of meteorological concern
- Elimination of terms on scaling considerations simplify mathematics and allows to eliminate and filter unwanted types of motions, like sound waves
- The following scales are assumed characteristics **for mid-latitude** synoptic systems

L	**$\approx$**	**10^6 m**	**horizontal length scale (~a nation)**
H	**$\approx$**	**10^4 m**	**vertical length scale (~troposphere)**
U	$\approx$	10 m s^{-1}	wind speed scale
W	$\approx$	0.01 m s^{-1}	vertical velocity scale
p	$\approx$	1000 hPa (10^5 Pa)	mean surface pressure
Δp	$\approx$	10 hPa (10^3 Pa)	pressure horizontal variation scale
$\Delta\rho/\rho$	$\approx$	10^{-2}	fractional density fluctuation
$T \approx L/U$	$\approx$	10^5 s	time scale

Scale analisis of horizontal NSEs

- It is possible to scale each term of each horizontal equation using previously defined scales and the definition of the Coriolis parameter (**mid-latitude**):

$$f_0 = 2\,\Omega \sin \phi_0 = 2\,\Omega \cos \phi_0 \approx 10^{-4}\ s^{-1} \qquad (\phi_0 = 45^\circ)$$

$$x-eq. \quad \frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v \sin\phi - 2\Omega w \cos\phi + f_{rx}$$

$$y-eq. \quad \frac{dv}{dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u \sin\phi + f_{ry}$$

$Scales$	U^2/L	$\frac{U^2}{a}$	$\frac{UW}{a}$	$\frac{\delta P}{\rho L}$	$f_0 U$	$f_0 W$	$\frac{\nu U}{L^2}$
(ms^{-2})	10^{-4}	10^{-5}	10^{-8}	10^{-3}	10^{-3}	10^{-6}	10^{-12}

The geostrophic approximation

From Ancient Greek γεω- (geō-, "earth-") + στροφή (strophḗ, "a turn, bend, twist")

- **Neglecting friction, gravitation and inertial acceleration (=stationary flow: du/dt=0) in the NS equations:**

$$\cancel{\frac{d\vec{U}}{dt}} = -2\vec{\Omega}\times\vec{U} - \frac{1}{\rho}\overrightarrow{\nabla p} + \cancel{\vec{g}^*} + \cancel{\vec{f}_r}$$

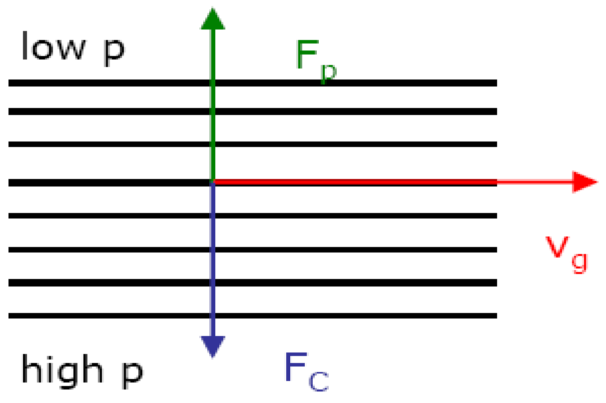
- **In cartesian coordinates (neglecting term with w):**

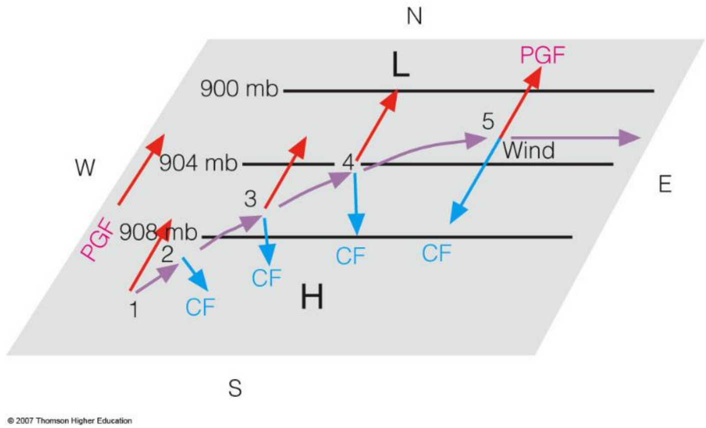
$$\left.\begin{aligned} \frac{1}{\rho}\frac{\partial p}{\partial x} &= 2\Omega v\cos\varphi \approx f_0 v & \rightarrow v_g &= \frac{1}{\rho f_0}\frac{\partial p}{\partial x} \\ \frac{1}{\rho}\frac{\partial p}{\partial y} &= -2\Omega u\sin\varphi \approx -f_0 u & \rightarrow u_g &= -\frac{1}{\rho f_0}\frac{\partial p}{\partial y} \end{aligned}\right\} \vec{u_g} = \vec{k}\times\frac{1}{\rho f_0}\overrightarrow{\nabla_H p}$$

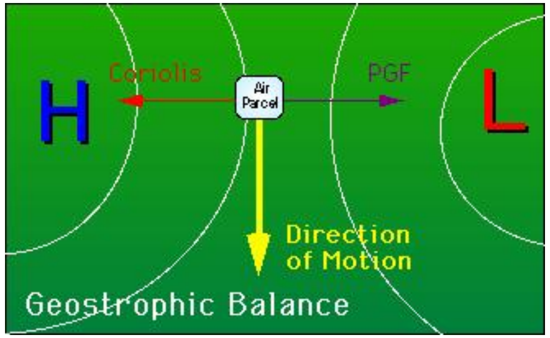
where the approximation $f_0=2\Omega\sin\varphi_0\approx 2\Omega\cos\varphi_0$ is valid at middle latitudes ($\varphi_0\approx 45^\circ$)

- **This equation defines the geostrophic wind as the wind produced by the balance between Coriolis and (horizontal) pressure gradient acceleration**
- **The above equation are <u>diagnostic</u>, not prognostic**

The geostrophic wind

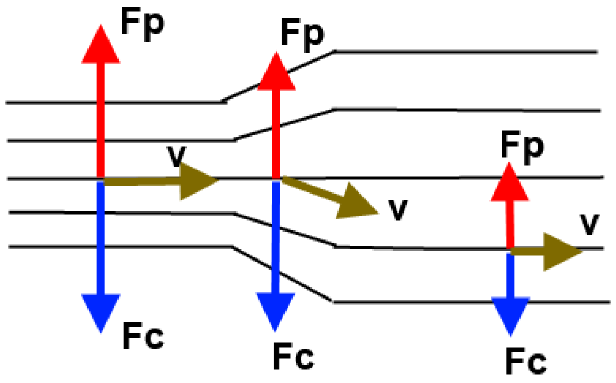


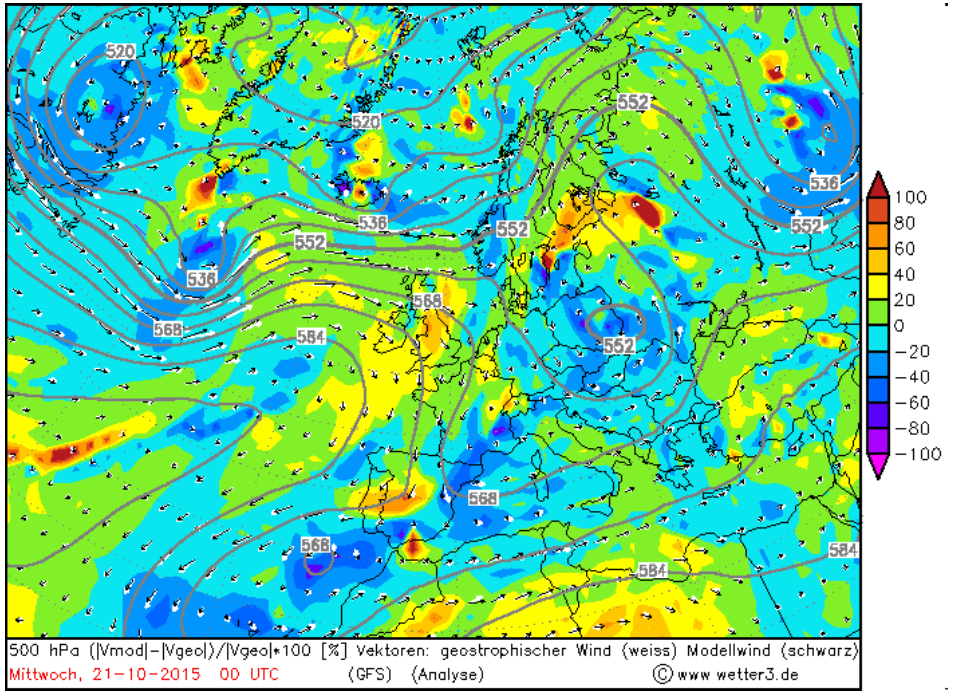




- The geostrophic wind is perpendicular to the pressure gradient
- In the free atmosphere, approximately 3 km above Earth surface, the inertial, curvature and friction accelerations are several order of magnitude lower than Coriolis and (horizontal) pressure gradient accelerations, and can be neglected
- Geostrophic wind is determined by the distribution of the surface pressure

Change of geostrophic wind



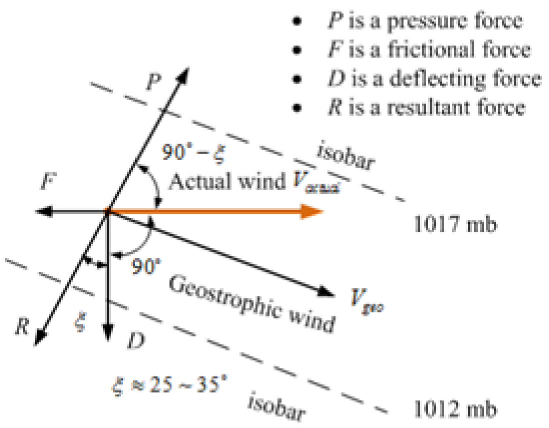


- **Example: consider case where geostrophic wind has been established**
- **Now move to a region with smaller p-gradient**
- **Wind speed becomes faster than during geostrophic balance**
- **Resulting force F_r leads to an acceleration of air parcel towards higher pressure**
- **Direction of Fc changes**
  - direction of F_r changes
  - change in v until new steady state is reached
- **Similar for case of stronger p-gradient**

The ageostrophic wind

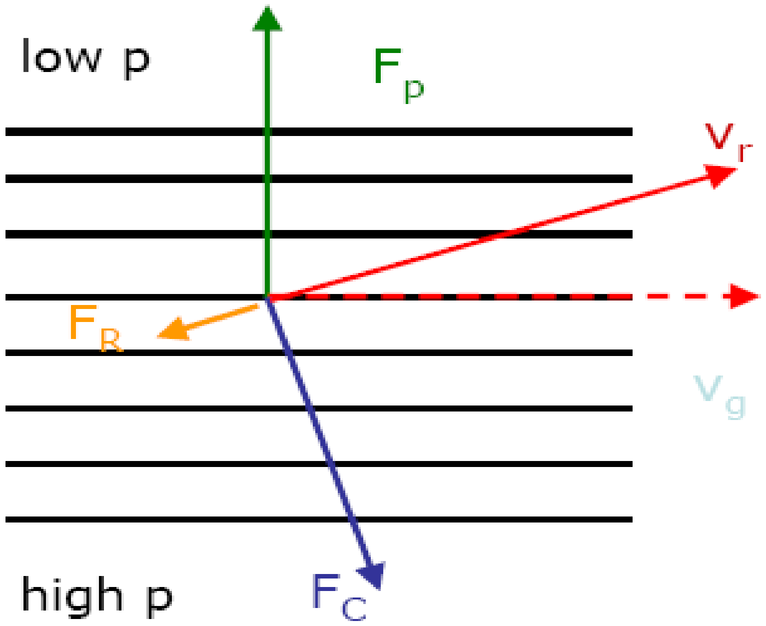
- **Geostrophic wind exists only for φ>10°, otherwise Coriolis acceleration ($\propto$sin φ) is too weak**
  - This means that, near Equator, Coriolis acceleration is weak: in case of strong minima/maxima, PGF is the largest force
    - ➔ at Equator there are NOT large minima/maxima
- **Pure geostrophic wind cannot reduce pressure gradient as it is parallel to isobars**
- **In the real atmosphere, friction exists always**
- **Molecular friction is only important in lowest few cm of the atmosphere, leading to the "no-slip boundary condition"**
- **Friction active in the troposphere is the "turbulent friction"**

Geostrophic wind with friction



- **Assume balance of PGF, Coriolis and (turbulent) friction:**

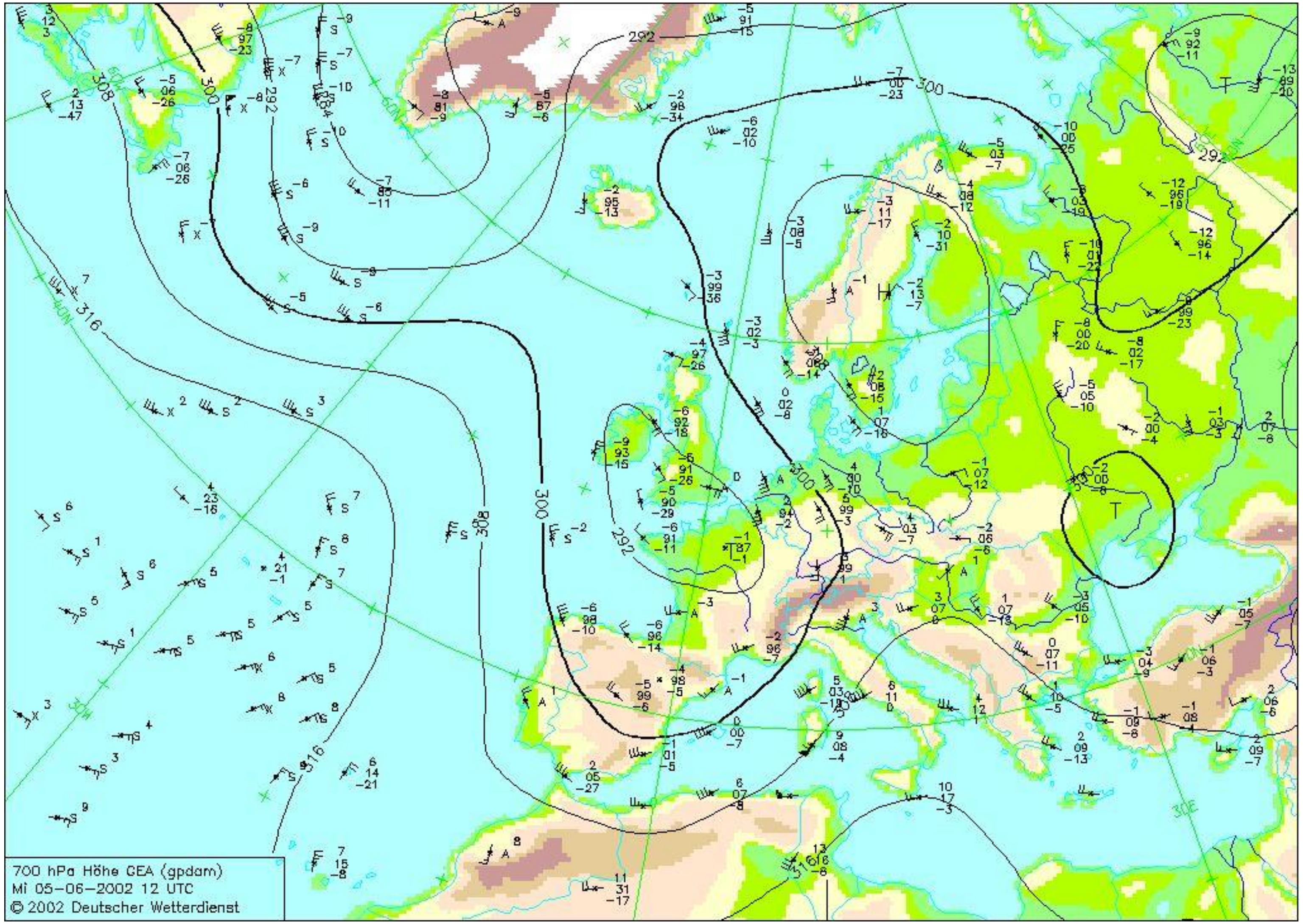
$$\cancel{\frac{d\vec{U}}{dt}} = -2\vec{\Omega}\times\vec{U} - \frac{1}{\rho}\overrightarrow{\nabla p} + \cancel{\vec{g}^{*}} + \vec{f}_r$$



- **Angle of wind velocity depends on surface friction, e.g.:**
  - ocean: 15°-30°
  - land: 25°-40°
  - "rough" land: 35°-50°

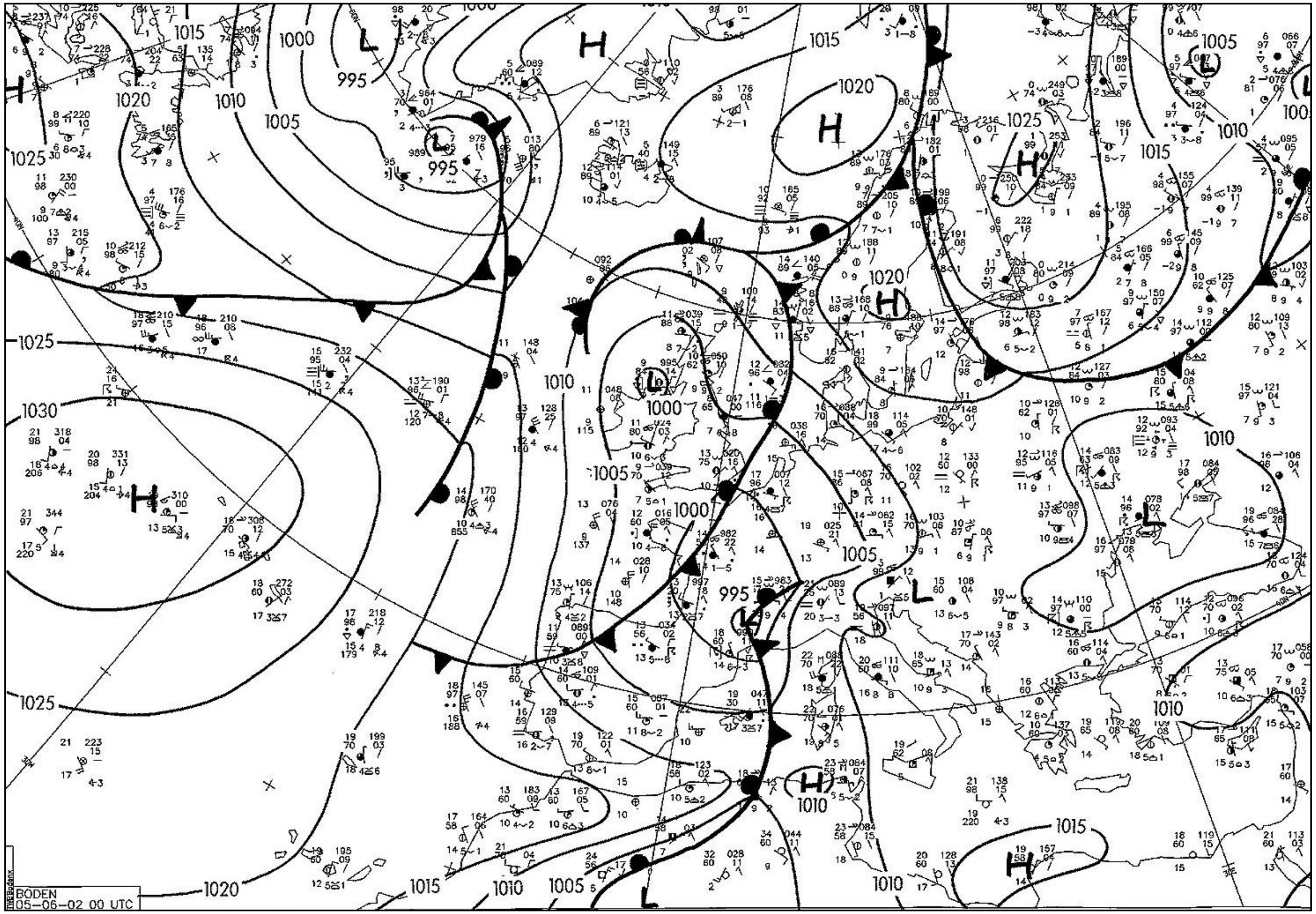
Geostrophic equilibrium @ 700 hPa

Geostrophic approximation is valid at 700 hPa or higher levels



Geostrophic equilibrium @ sfc

Geostrophic approximation is less valid at the surface or lower levels



Approximate prognostic equation

- In these equations, acceleration term (small) is given by the difference between two larger terms → large errors in its determination
- Measure of magnitude of acceleration (inertial force) compared with Coriolis force is given by the **Rossby number Ro=U/f_0L**

$$\frac{inertial\ term}{Coriolis\ term} = \frac{U^2/L}{f_0 U} = \frac{U}{f_0 L} = Ro$$

- The physical meaning of this fact is that the geostrophic equilibrium represents a sort of "attractor" and the deviations from this equilibrium, produced by pressure and/or density fluctuations, generate geostrophic re-adjustment accelerations which are one order of magnitude lower and which guide the atmosphere to another geostrophic state

Quasi-geostrophic approximation

- The geostrophic wind is determined by the pressure field
- It represent an idealization of the real wind, good at synoptic scale
- To look at a most real situation, in which wind field evolve in time, it is necessary to consider inertial terms:

$$\frac{d\vec{U}}{dt} = -2\vec{\Omega}\times\vec{U} - \frac{1}{\rho}\overrightarrow{\nabla p} + \cancel{\vec{g}} + \cancel{\vec{f}_r}$$

or:

$$\frac{du}{dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \frac{dv}{dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

- The two additional non-stationary terms with time derivatives represent the re-adjustment between the terms of Coriolis and of pressure gradient during the time evolution
- These are the simplest prognostic equations of atmospheric motions

Quasi-geostrophic approximation

- The importance of inertial terms (accelerations) with respect to the Coriolis terms is expressed by the ratio between the respective scale accelerations
- The characteristic scale for the inertial accelerations is suggested by the advective part, and its magnitude order is about U^2/L, U and L being, respectively, the velocity and geometric scales
- The characteristic scale for the Coriolis term is instead f_oU
- Then, the ratio between U^2/L and f_oU, i.e. U/f_oL, defines a dimensionless number known as Rossby number, which express the limit of validity of the geostrophic approximation
- The geostrophic approximation is better as smaller is the value of Ro

Approximate prognostic equation

- Retaining only the two biggest terms, it is possible to write the geostrophic balance. This allows the definition of geostrophic wind as balance between Coriolis and pressure gradient forces:

$$-fv \approx -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad fu \approx -\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \mathbf{V}_g \equiv \mathbf{k} \times \frac{1}{\rho f}\nabla p$$

- However, this equation is **diagnostic** (no reference to time). To obtain prediction equation it is necessary to retain also acceleration term. The resulting approximate horizontal equations can be written as:

$$\frac{du}{dt} = fv - \frac{1}{\rho}\frac{\partial p}{\partial x} = f\left(v - v_g\right)$$

$$\frac{dv}{dt} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y} = -f\left(u - u_g\right)$$

$$\frac{d\vec{u}}{dt} + f\vec{k} \times \vec{u} = \frac{1}{\rho}\nabla p$$

Scale analisis of vertical equation

- By scaling the vertical equation, we arrive to:

$$z-eq. \quad \frac{dw}{dt} - \frac{u^2+v^2}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + 2\Omega u\cos\phi + g + F_{rz}$$

$$Scales \quad UW\!/\!L \quad \frac{U^2}{a} \quad \frac{\Delta p}{\rho H} \quad f_0 U \quad g \quad \frac{\nu W}{H^2}$$

$$(ms^{-2}) \quad 10^{-7} \quad 10^{-5} \quad 10 \quad 10^{-3} \quad 10 \quad 10^{-11}$$

- The biggest terms (gravity and PGF) give the hydrostatic approximation:

$$\frac{1}{\rho_0}\frac{dp_0}{dz} \equiv -g$$

Hydrostatic approximation

- In the equation for the vertical speed component [dw/dt] of NSEs, the dominant terms at synoptic scale are gravity and PGF
- Their values are about 8 order of magnitude bigger than those of inertial term
- These two terms must thus balance each other:

$$\frac{1}{\rho}\frac{\partial p}{\partial z} \approx -g$$

- In other words, the atmospheric pressure must be, in synoptic motions, in each point and moment, equal to the weight of the unit air column above it
- However this condition may not be sufficient: it is necessary to verify that this condition remains valid even in presence, at synoptic scale, of perturbed horizontal velocity fields – i.e. of horizontal pressure fields at several levels

Hydrostatic approximation

- To demonstrate that the condition [dw/dt<<g] is valid at synoptic scale, a perturbative analysis is needed
- Let's define vertical pressure and density fields $\{p_0(z)\}$ and $\{\rho_0(z)\}$ as horizontal averages in x, y and t at each heigth z, which satisfy the exact hydrostatic equilibrium:

$$\frac{1}{\rho_0}\frac{\partial p_0}{\partial z} \approx -g$$

- Let's decompose the fields:

$$p(x,y,z,t) = p_0(z) + p'(x,y,z,t) \qquad \rho(x,y,z,t) = \rho_0(z) + \rho'(x,y,z,t)$$

- Considering $\rho'/\rho_0 \approx 10^{-2}$,

$$\frac{1}{(\rho_0+\rho')} = \frac{(\rho_0-\rho')}{(\rho_0+\rho')(\rho_0-\rho')} = \frac{(\rho_0-\rho')}{(\rho_0^2-\rho'^2)} \cong \frac{1}{\rho_0^2}(\rho_0-\rho') = \frac{1}{\rho_0}\left(1-\frac{\rho'}{\rho_0}\right)$$

so we get:

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} - g = -\frac{1}{(\rho_0+\rho')}\frac{\partial(p_0+p')}{\partial z} - g \cong \frac{1}{\rho_0}\left(\frac{\rho'}{\rho_0}\frac{dp_0}{dz} - \frac{\partial p'}{\partial z}\right) = -\frac{1}{\rho_0}\left(\rho' g + \frac{\partial p'}{\partial z}\right)$$

Hydrostatic approximation

- **The order of magnitude of the last two term is 10^{-1} (ms^{-2})**
- ***This result shows that the scale accelerations associated to the two perturbed terms of pressure and density are several order of magnitude larger than those associated to other terms (in particular the inertial one dw/dt) and must thus be equal and opposite***

$$-\frac{1}{\rho}\frac{\partial p}{\partial z}-g=\frac{1}{\rho_0}\left(-\frac{dp_0}{dz}-\rho_0 g\right)-\frac{1}{\rho_0}\left(\rho' g+\frac{\partial p'}{\partial z}\right)=0-\frac{1}{\rho_0}\left(\rho' g+\frac{\partial p'}{\partial z}\right)$$

- **Being the first bracket at right identically null, thus also the perturbed part has the same order of magnitude than the mean field**
- **Thus also the perturbed pressure field is in hydrostatic equilibrium, with an approximation of 1 part on 10^3**
- **This means that, at synoptic scale, the vertical accelerations are negligible and, consequently, vertical velocities cannot be deduced by the vertical component of NSEs, which has been substituted by static equation**

Isobaric coordinates: geostrophic wind

- Using the positions $\partial p/\partial z = -\rho g$ and $g\,dz = d\Phi$, geostrophic wind becomes:

$$\left.\begin{aligned} v_g &= \frac{1}{\rho f_0}\frac{\partial p}{\partial x} = -\frac{1}{f_0}\frac{\partial \Phi}{\partial x} \\ u_g &= -\frac{1}{\rho f_0}\frac{\partial p}{\partial y} = \frac{1}{f_0}\frac{\partial \Phi}{\partial y} \end{aligned}\right\} \overrightarrow{u_g} = \vec{k} \times \frac{1}{f}\overrightarrow{\nabla\Phi}$$

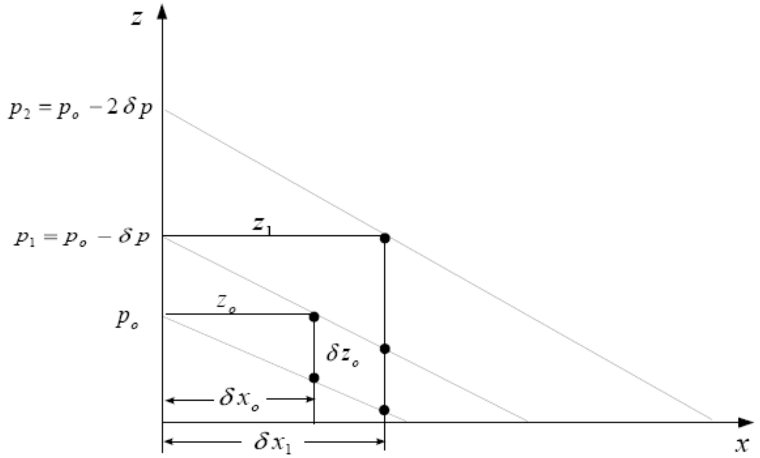
- and approximate equation for horizontal motions:

$$\frac{d\vec{u}}{dt} + f\,\vec{k} \times \vec{u} = \frac{\overrightarrow{\nabla p}}{\rho} = -\overrightarrow{\nabla\Phi}$$

- In both cases, density not appears explicitly in equations in isobaric coordinates, thus the geostrophic law is explicitly independent from the height

Isobaric coordinates: barotropy

- **When isobaric surfaces are parallel (barotropic atmosphere), geostrophic wind is constant with height**
- ***At higher quotes density decreases and pressure gradients decrease too: vertical pressure gradients vary with height***
- **However, in isobaric coordinates, as density do not appear in the equation, distance between isobaric surfaces does not change, and vertical geopotential gradients between isobaric surfaces are equal at all levels:**
  - in isobaric coordinates, iso-geopotential surfaces are not only parallel as isobaric surfaces in geometric coordinates, but also at equal distance



$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial z}\left(\frac{\delta z}{\delta x}\right)_p = g\left(\frac{\delta z}{\delta x}\right)_p$$

Isobaric coordinates: geostrophy

- Moreover, in the motion in which f can be considered constant and wind is quasi geostrophic, it is:

$$\overrightarrow{\nabla_p} \cdot \overrightarrow{u_g} = \overrightarrow{\nabla_p} \cdot \left(\vec{k} \times \frac{1}{f} \overrightarrow{\nabla\Phi} \right) = \frac{1}{f} \overrightarrow{\nabla_p} \cdot \left(\vec{k} \times \overrightarrow{\nabla\Phi} \right) = \frac{1}{f} \left[\overrightarrow{\nabla\Phi} \cdot \left(\vec{\nabla} \times \vec{k} \right) - \vec{k} \cdot \left(\vec{\nabla} \times \overrightarrow{\nabla_p \Phi} \right) \right] = 0$$

  where it has been used the identity:

$$\vec{\nabla} \cdot \left(\vec{A} \times \vec{B} \right) = \left[\vec{B} \cdot \left(\vec{\nabla} \times \vec{A} \right) - \vec{A} \cdot \left(\nabla \times \vec{B} \right) \right] = 0$$ and $\vec{\nabla} \times \vec{k} = 0$ $\quad \vec{\nabla} \times \overrightarrow{\nabla\Phi} = 0$

- The meaning of this equation is that geostrophic wind over isobaric surfaces is a nondivergent vector, i.e. a solenoidal vector, without sources or sinks
- This result has been obtained using isobaric coordinates under the only hypothesis of f constant (without need of ρ constant)