Vorticity and Potential Vorticity (PV)

Textbooks and web sites references for this lecture:

- James R. Holton, An Introduction to Dynamic Meteorology, Academic Press, 1992, ISBN 0-12-354355-X
- A. Longhetto – Dispense di Fisica dell'atmosfera
- Physic der atmosphäre - Institut für Umweltphysik - Universität Heidelberg
- Adrian M. Tomkins – Atmospheric Physics – ictp_atmospheric_physics.beamer.pdf

Circulation and its theorem

- Definition of circulation around a closed contour in a fluid: $C = \oint_{\gamma} \overline{V} \bullet d\overline{r} = \int_{S} \overline{\zeta} \bullet \overline{n}\, dS$
- Using (frictionless) Navier-Stokes equations: $\frac{d\overline{V}}{dt} = -\frac{\overline{\nabla p}}{\rho} - 2\overline{\Omega} \times \overline{V} + \overline{g}$

$$\frac{dC}{dt} = -\oint_{\gamma} \frac{\overline{\nabla p}}{\rho} \bullet d\overline{r} - 2\oint_{\gamma} \overline{\Omega} \times \overline{V} \bullet d\overline{r}$$ is the the derivative of circulation is:

and remembering that $\overline{V_a} = \overline{V} + 2\overline{\Omega} \times \overline{R}$ and $C_a = C + 2\Omega$

(definitions of absolute variables, inclusive of local velocity and Earth's angular velocity)

the derivative of absolute circulation can be defined as:

$$\frac{dC_a}{dt} = -\oint_{\gamma} \frac{\overline{\nabla p}}{\rho} \bullet d\overline{r} = -\oint_{\gamma} \frac{dp}{\rho}$$

from which we see that, in barotropic atmosphere, circulation is null

The concept of vorticity

- Apart from the conservation of energy and momentum, the angular momentum is a conserved quantity in a dynamical system
- In atmospheric physics, the conservation of angular momentum is expressed as the conservation of the vortex strength of the wind vector field
- Different definitions of the vortex strength exist

Vorticity

- This vectorial field can be considered as the microscopic measure of rotation in a fluid, and is defined as the curl of velocity:

$$\overline{\omega} = \overline{\nabla} \times \overline{v}$$

- The components of ω are:

$$\vec{\omega} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

- In meteorology, the curl of the wind vector field is only important in the horizontal, since the vertical extent of the atmosphere is very small, thus only vertical components of ω is relevant:

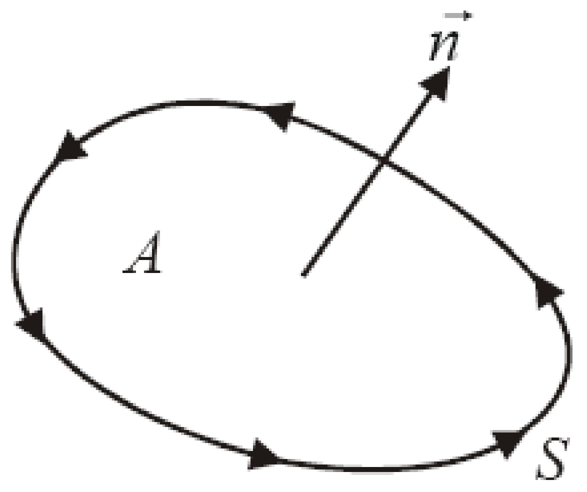
$$\zeta = \vec{k} \cdot \vec{\omega} = \vec{k} \cdot \left(\vec{\nabla} \times \vec{v} \right)$$

Vorticity and circulation

- The curl of a vector field perpendicular to a given surface A with normal vector is related to the circulation Z of the vector field (i.e., the closed path integral along the border S of A):

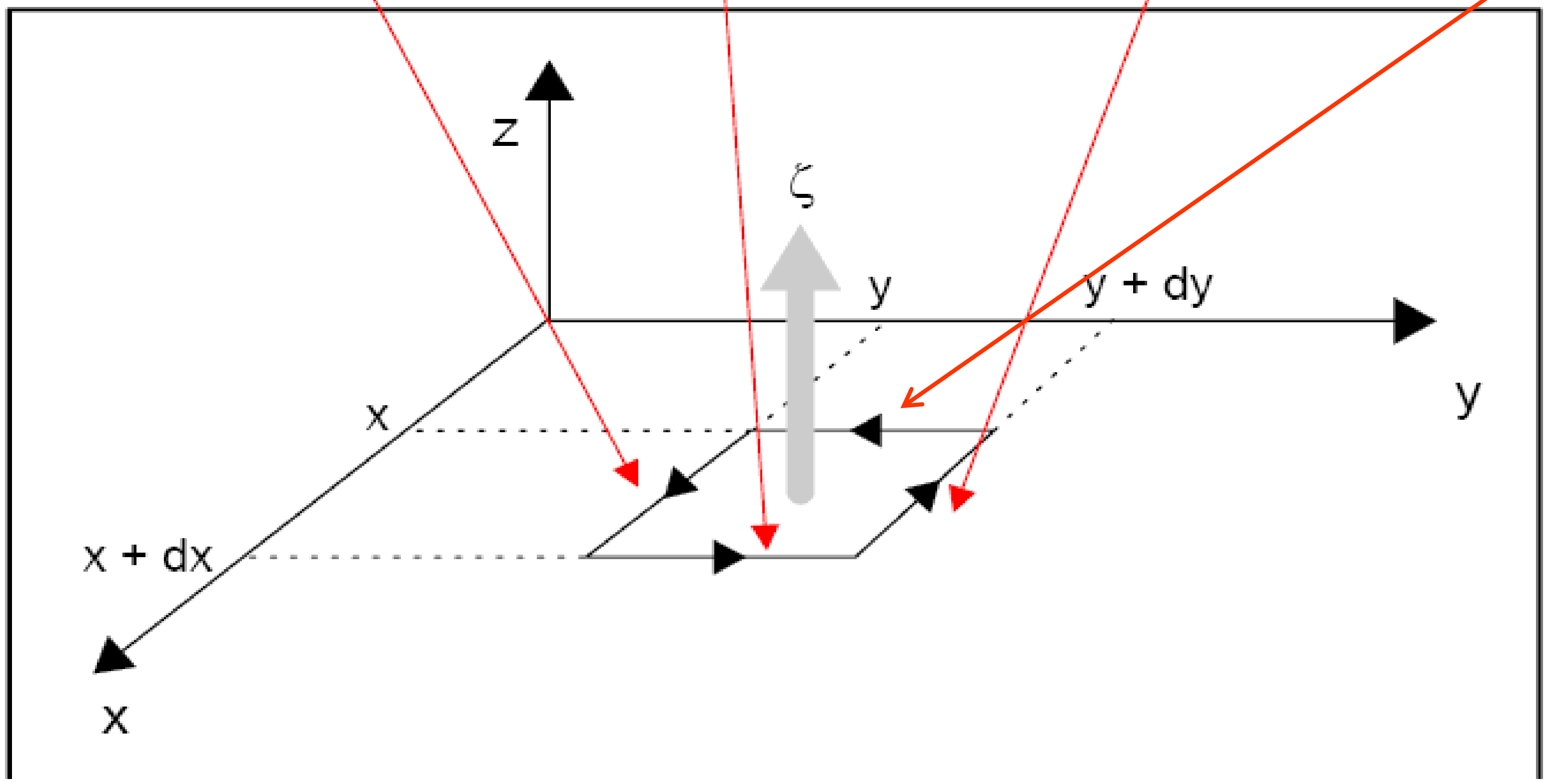
$$\Gamma = \oint_A \vec{v} \cdot d\vec{s}$$

$$\left(\nabla \times \vec{v}\right)_{\vec{n}} = \lim_{A \to 0} \frac{\Gamma(A)}{A} = \frac{d}{dA} \oint_{S(A)} \vec{v} \cdot d\vec{s}$$



Vorticity and circulation

- Circulation in Cartesian coordinates:
- $$dG = \underbrace{u(y)dx}_{1} + \underbrace{v(x+dx)dy}_{2} - \underbrace{u(y+dy)dx}_{3} - \underbrace{v(x)dy}_{4}$$



Vorticity and circulation

- With dA=dxdy the above equation becomes:

$$d\Gamma = u(y)\,dx + v(x+dx)\,dy - u(y+dy)\,dx - v(x)dy =$$

$$= u(y)\,dx - v(x)\,dy + \left[v(x) + \frac{\partial v}{\partial x}dx\right]dy - \left[u(y) + \frac{\partial u}{\partial y}dy\right]dx =$$

$$= \left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]dx\,dy$$

$$\zeta = \frac{d\Gamma}{dA} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

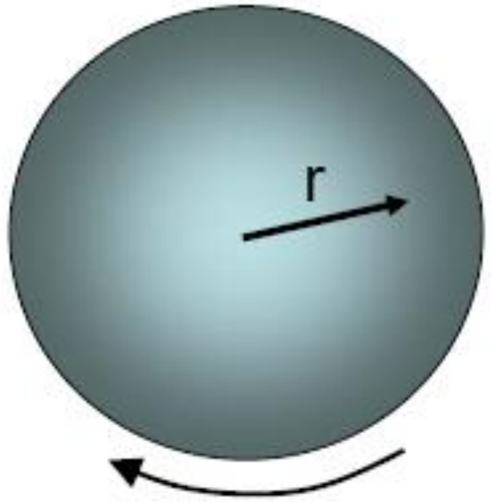
- Note that these findings are a special case of the Stokes theorem:

$$\oint_{S(A)} \vec{v} \cdot d\vec{s} = \int_A \left(\nabla \times \vec{v}\right) d\vec{A}$$

- Sign of ζ:
  - Positive for counterclockwise rotation
  - Negative for clockwise rotation

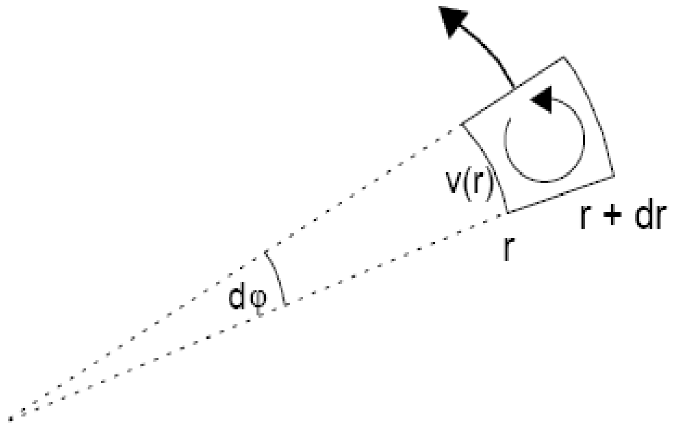
Vorticity of a Rigid Rotator

- Velocity at distance r from the axis:
- $v = \omega r$
- Circulation: $\Gamma = \oint_{S(A)} \vec{v} \cdot d\vec{s} = 2\pi r v = 2\pi r^2 \omega$
- Thus, the vorticity is: $\zeta = \frac{d\Gamma}{dA} = \frac{\Gamma}{\pi r^2} = \frac{2\pi r^2 \omega}{\pi r^2} = 2\omega$



- → The vorticity of a rigid rotator is twice its angular velocity
- Example: High pressure system, R = 500 km, $v=10\ ms^{-1}$ → $\omega = v/r \approx 10/(5 \cdot 10^5) = 2 \cdot 10^{-5}\ s^{-1}$ → $\zeta = 2\ \omega = 4 \cdot 10^{-5}\ s^{-1}$

Vorticity of a curved trajectory



- General case of a curved trajectory
- Radius of curvature function of ds:
- r ⊥ v: d**v•r**=0
- Circulation around infinitesimal area: dA= (r dϕ) (dr)

$$d\Gamma = -v(r)\,r\,d\phi + v(r+dr)(r+dr)\,d\phi = -v(r)\,r\,d\phi + \left(v(r) + \frac{\partial v}{\partial r}dr\right)(r+dr)\,d\phi =$$

$$= \cancel{-v(r)\,r\,d\phi} + \cancel{v(r)\,r\,d\phi} + v(r)\,dr\,d\phi + \frac{\partial v}{\partial r}dr\,r d\phi + \cancel{\frac{\partial v}{\partial r}dr\,dr\,d\phi} =$$

$$\cong v(r)\,dr\,d\phi + \frac{\partial v}{\partial r}r\,dr\,d\phi$$

- Dividing by the infinitesimal area dA:

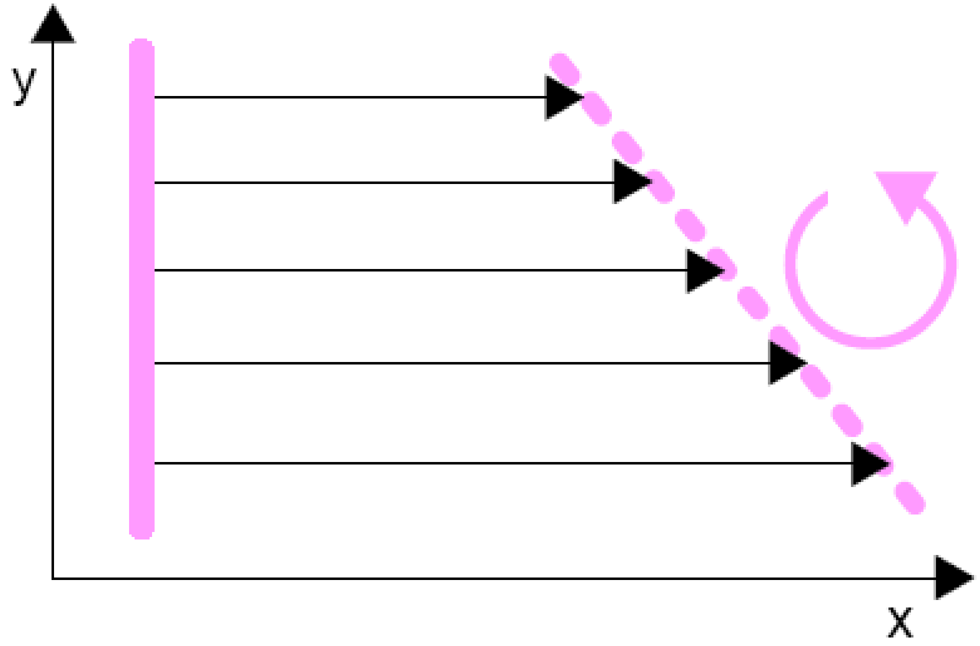
$$\zeta = \frac{d\Gamma}{dA} = \frac{v(r)\,r\,d\phi + \frac{\partial v}{\partial r}r\,dr\,d\phi}{r\,dr\,d\phi} = \frac{v(r)}{r} + \frac{\partial v}{\partial r}$$

In case of a rigid rotator, v=ωr so dv/dr=ω and ζ=ω+ω=2ω as said before

Vorticity of shear winds

- Let's assume flow (wind) in x-direction, for simplicity
- Thus v=0 and $\partial v/\partial x=0$
- Thus the vorticity is:

$$\zeta = \left(\nabla \times \vec{v}\right)_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial y}$$



- A straight line connecting different air parcels in y-direction will rotate, due to wind shear

The absolute vorticity

- The dynamics of the atmosphere is described in a rotating coordinate system
- The Earth is a rigid rotator with angular velocity Ω
- Local (planetary) vorticity in z-direction at latitude ϕ is given by the Coriolis parameter
- $f = 2\ \Omega \sin \phi$
  - (see previously discussed vorticity of a rigid rotator)
- Thus the absolute vorticity ω_a (and its vertical component ζ_a) of the wind field is the sum of the relative vorticity ω (and its vertical component ζ), measured relative to the terrestrial coordinate system, and the Coriolis parameter (where f is the vertical component):

$$\zeta_a = \zeta + f \equiv \left(\vec{\nabla} \times \vec{v}\right)_z + 2\Omega \sin \phi$$

Vorticity

From the equations of motion in a rotating system with angular velocity $\mathbf{\Omega}$, neglecting frictional forces:

$$\frac{d\vec{v}}{dt} = -\frac{\overrightarrow{\nabla p}}{\rho} - 2\vec{\Omega}\times\vec{v} - g\vec{k}$$

where:

$$\frac{d\vec{v}}{dt} = \frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}$$

The advective term can be rewritten as: $(\vec{v}\cdot\vec{\nabla})\vec{v} = \vec{\nabla}(\frac{1}{2}v^2) + (\vec{\nabla}\times\vec{v})\times\vec{v}$

Then by applying the curl operator $\nabla\times$ to the above momentum equation:

$$\frac{\partial}{\partial t}(\nabla\times\vec{v}) + \cancel{\nabla\times[\nabla(\tfrac{1}{2}v^2)]} + \nabla\times[(\nabla\times\vec{v})\times\vec{v}] =$$
$$\cancel{-\frac{1}{\rho}\nabla\times(\nabla p)} - \nabla(\frac{1}{\rho})\times\nabla p - 2\nabla\times(\vec{\Omega}\times\vec{v}) \cancel{- \nabla\times(\nabla\phi)}$$

where terms vanish because the curl of a gradient vanishes $\nabla\times(\nabla a) = 0,$

$g\hat{k} = \nabla gz = \nabla\phi$ and using: $\nabla\times(a\vec{A}) = a\nabla\times\vec{A} + \nabla a\times\vec{A}.$

The vorticity equation

- Being the absolute vorticity equal to $\omega_a = \omega + 2\Omega$, it is: $\frac{\partial \vec{\omega}}{\partial t} + \vec{\nabla} \times (\vec{\omega} \times \vec{v}) + \vec{\nabla} \times (2\vec{\Omega} \times \vec{v}) = -\overrightarrow{\nabla \alpha} \times \overrightarrow{\nabla p}$

  where α is the specific volume $1/\rho$

- Also, considering that: $\frac{\partial}{\partial t}\Omega = 0,$ it is $\frac{\partial \overrightarrow{\omega_a}}{\partial t} = \frac{\partial \vec{\omega}}{\partial t} + 2\frac{\partial \vec{\Omega}}{\partial t} = \frac{\partial \vec{\omega}}{\partial t}$

$$\frac{\partial \overrightarrow{\omega_a}}{\partial t} + \nabla \times (\overrightarrow{\omega_a} \times \vec{v}) = -\nabla \alpha \times \nabla p$$

- Then by applying the standard vector relationship:

$$\nabla \times (\vec{A} \times \vec{B}) = \vec{A}\nabla \cdot \vec{B} - \vec{B}\nabla \cdot \vec{A} + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B},$$

$$\nabla \times (\overrightarrow{\omega_a} \times \vec{v}) = \overrightarrow{\omega_a}\nabla \cdot \vec{v} - \vec{v}\nabla \cdot \overrightarrow{\omega_a} + (\vec{v} \cdot \nabla)\overrightarrow{\omega_a} - (\overrightarrow{\omega_a} \cdot \nabla)\vec{v}$$

  where the 2nd RHS term vanishes because it contains the divergence of a curl (the vorticity).

- Re-arranging terms (the 3rd RHS is the vorticity advection):

$$\frac{d\overrightarrow{\omega_a}}{dt} + \overrightarrow{\omega_a}\nabla \cdot \vec{v} - (\overrightarrow{\omega_a} \cdot \nabla)\vec{v} = -\nabla \alpha \times \nabla p$$

The vorticity equation

- We use now the continuity equation $\frac{1}{\rho}\frac{d\rho}{dt} + \nabla \cdot \vec{v} = 0$ multiplied by $-\frac{\overrightarrow{\omega_a}}{\rho}$,

  and the vorticity eqn. multiplied by 1/ρ:

$$\left\{\begin{array}{c} -\frac{\overrightarrow{\omega_a}}{\rho^2}\frac{d\rho}{dt} - \frac{\overrightarrow{\omega_a}}{\rho}\vec{\nabla}\cdot\vec{v} = 0 \\ \frac{1}{\rho}\frac{d\overrightarrow{\omega_a}}{dt} + \frac{\overrightarrow{\omega_a}}{\rho}\vec{\nabla}\cdot\vec{v} - \frac{1}{\rho}\left(\overrightarrow{\omega_a}\cdot\vec{\nabla}\right)\vec{v} = -\frac{1}{\rho}\overrightarrow{\nabla\alpha}\times\overrightarrow{\nabla p} \end{array}\right\} - \frac{\overrightarrow{\omega_a}}{\rho^2}\frac{d\rho}{dt} + \frac{1}{\rho}\frac{d\overrightarrow{\omega_a}}{dt} - \frac{1}{\rho}\left(\overrightarrow{\omega_a}\cdot\vec{\nabla}\right)\vec{v} = -\frac{1}{\rho}\overrightarrow{\nabla\alpha}\times\overrightarrow{\nabla p}$$

- Summing and multipliying by ρ:

$$\rho\frac{d}{dt}\left(\frac{\overrightarrow{\omega_a}}{\rho}\right) = \underbrace{\left(\overrightarrow{\omega_a}\cdot\vec{\nabla}\right)\vec{v}}_{\textit{stretching} \text{ and } \textit{twisting} \text{ terms}} - \underbrace{\overrightarrow{\nabla\alpha}\times\overrightarrow{\nabla p}}_{\textit{baroclinic} \text{ or } \textit{solenoidal} \text{ term}}$$

The vorticity equation: vertical component

- This equation can be rewritten as:

$$\rho \frac{d}{dt}\left[\frac{1}{\rho}\begin{pmatrix}\omega_{a,x}\\ \omega_{a,y}\\ \omega_{a,z}\end{pmatrix}\right] = \left(\omega_{a,x}\frac{\partial}{\partial x} + \omega_{a,y}\frac{\partial}{\partial y} + \omega_{a,z}\frac{\partial}{\partial z}\right)\begin{pmatrix}u\\ v\\ w\end{pmatrix} - \begin{pmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial \alpha}{\partial x} & \frac{\partial \alpha}{\partial y} & \frac{\partial \alpha}{\partial z}\\ \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} & \frac{\partial p}{\partial z}\end{pmatrix}$$

- The vertical component of total (i.e. absolute) vorticity is the most relevant for cyclones/anticyclones, and can be written as:

$$\rho \frac{d}{dt}\left(\frac{\omega_{a,z}}{\rho}\right) = \underbrace{\omega_{a,x}\frac{\partial w}{\partial x} + \omega_{a,y}\frac{\partial w}{\partial y}}_{\text{twisting or tilting terms}} + \underbrace{\omega_{a,z}\frac{\partial w}{\partial z}}_{\text{stretching term}} - \underbrace{\nabla_H \alpha \times \nabla_H p}_{\text{baroclinic or solenoidal term}}$$

The vorticity equation: vertical component

- Taking into account the exact values of the absolute vorticity components, and neglecting w in the definition of vorticity components (good assumption in large-scale meteorology), equation simplifies in:

$$\rho \frac{d}{dt}\left(\frac{\omega_{a,z}}{\rho}\right) = \omega_{a,x}\frac{\partial w}{\partial x} + \omega_{a,y}\frac{\partial w}{\partial y} + \omega_{a,z}\frac{\partial w}{\partial z} - \nabla_H \alpha \times \nabla_H p$$

$$\omega_{a,z} = \zeta_a = \zeta + f = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f$$

$$\omega_{a,x} = \omega_x \cong -\frac{\partial v}{\partial z}$$

$$\omega_{a,y} = \omega_y \cong \frac{\partial u}{\partial z}$$

$$\rho \frac{d}{dt}\left(\frac{\zeta + f}{\rho}\right) = -\frac{\partial v}{\partial z}\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial y} + (\zeta + f)\frac{\partial w}{\partial z} - \nabla_H \alpha \times \nabla_H p$$

Scale analysis of the vert. comp. of vort. eq.

- In **mid-latitude synoptic scale** dynamic meteorology, typical magnitudes are:

$U \approx 10$ m s^{-1}	wind speed scale
$W \approx 0.01$ m s^{-1}	vertical velocity scale
$L \approx 10^6$ m	horizontal length scale
$H \approx 10^4$ m	vertical length scale
$p \approx 1000$ hPa (10^5 Pa)	mean surface pressure
$\Delta p \approx 10$ hPa (10^3 Pa)	pressure horiz. var. scale

$\Delta p/p \approx 10^{-2}$	fract. press. fluctuation
$\rho \approx 1$ kg m^3	mean surface density
$\Delta\rho/\rho \approx 10^{-2}$	fract. density fluctuation
$T \approx L/U \approx 10^5$ s	time scale
$f_0 \approx 10^{-4}$ s^{-1}	mid lat. Coriolis param.
$\beta = df/dy \approx 10^{-11}$ m^{-1} s^{-1}	beta parameter

- With the above <u>scales</u>:
  - the relative vorticity $\zeta = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} \approx U/L \approx 10^{-5}$ s^{-1}
  - the ratio ζ/f_0 $\approx Ro = U/f_0L \approx 10^{-1}$

- Thus relative vorticity, at this scale, is smaller than planetary vorticity

Scale analysis of the vert. comp. of vort. eq. (2)

- Total derivative with respect to time can be splitted as:

$$\rho \frac{d}{dt}\left(\frac{\zeta + f}{\rho}\right) = \rho\left[\frac{\partial}{\partial t} + \overrightarrow{v_H} \cdot \vec{\nabla} + w\frac{\partial}{\partial z}\right]\left(\frac{\zeta + f}{\rho}\right) =$$

$$= \frac{\partial \zeta}{\partial t} + \cancel{\frac{\partial f}{\partial t}} + \overrightarrow{v_H} \cdot \vec{\nabla}\zeta + \overrightarrow{v_H} \cdot \vec{\nabla} f + w\frac{\partial \zeta}{\partial z} + \cancel{w\frac{\partial f}{\partial z}} - \frac{\zeta + f}{\rho}\left[\cancel{\frac{\partial \rho}{\partial t}} + \overrightarrow{v_H} \cdot \vec{\nabla}\rho + w\frac{\partial \rho}{\partial z}\right] =$$

$$\cong \frac{\partial \zeta}{\partial t} + \overrightarrow{v_H} \cdot \vec{\nabla}\zeta + v\frac{\partial f}{\partial y} + w\frac{\partial \zeta}{\partial z} - \frac{\overrightarrow{v_H} \cdot \vec{\nabla}\rho}{\rho}(\zeta + f) - \frac{w}{\rho}\frac{\partial \rho}{\partial z}(\zeta + f)$$

- where all derivatives of f are null (except that of y), and where we have neglected the term containing $\partial r/\partial t$
- Let now examine these terms one by one

Scale analysis of the vert. comp. of vort. eq. (3): LHS

The LHS (total derivative) of the vertical component of the vorticity equation splits into:

- $\frac{\partial \zeta}{\partial t},\ \vec{V}_H \cdot \nabla_H \zeta$ $\approx U^2/L^2 \approx 10^{-10}\ s^{-2}$
- $w\frac{\partial \zeta}{\partial z}$ $\approx WU/(LH) \approx 10^{-11}\ s^{-2}$
- $v\frac{\partial f}{\partial y}$ $\approx U\beta \approx 10^{-10}\ s^{-2}$
- $-\frac{\zeta}{\rho}\vec{V}_H \cdot \nabla_H \rho$ $\approx U^2(\Delta\rho/\rho)/L^2 \approx 10^{-12}\ s^{-2}$ $\quad -\frac{f}{\rho}\vec{V}_H \cdot \nabla_H \rho$ $\approx f_0 U\ (\Delta\rho/\rho)/L \approx 10^{-11}\ s^{-2}$
- $-\frac{\zeta}{\rho} w\frac{\partial \rho}{\partial z}$ $\approx UW(\Delta\rho/\rho)/HL \approx 10^{-13}\ s^{-2}$ $\quad -\frac{f}{\rho} w\frac{\partial \rho}{\partial z}$ $\approx f_0 W\ (\Delta\rho/\rho)/H \approx 10^{-12}\ s^{-2}$

Scale analysis of the vert. comp. of vort. eq. (4): RHS

The RHS (total derivative) scaling of the vertical component of the vorticity equation splits into:

- $\frac{\partial v}{\partial z}\frac{\partial w}{\partial x} \approx \frac{\partial u}{\partial z}\frac{\partial w}{\partial y}$ $\approx$ (U/H)(W/L) $\approx 10^{-11}\ s^{-2}$
- $\zeta\frac{\partial w}{\partial z} \approx$ U W/(HL) $\approx 10^{-11}\ s^{-2}$ $\quad f\frac{\partial w}{\partial z} \approx f_0$ W/H $\approx 10^{-10}\ s^{-2}$
- $\nabla_H \alpha \times \nabla_H p$ $\approx (\Delta\rho/\rho)\ \Delta p\ /\ (\rho L^2) \approx 10^{-11}\ s^{-2}$
- The only terms remained ($\propto 10^{-10}\ s^{-2}$) are:

$$\frac{\partial \zeta}{\partial t} + \overrightarrow{v_H}\cdot\vec{\nabla}\zeta + v\frac{\partial f}{\partial y} \cong f\frac{\partial w}{\partial z}$$

Results of scale analysis

- The stretching term is therefore the most important for mid-latitude synoptic scale dynamics. In this case, it is:

$$\frac{\partial w}{\partial z} \approx -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

  and therefore the horizontal divergence is ≈ W/H ≈ 10^{-6} s^{-1}, i.e. about one order of magnitude smaller than individual terms that are ≈ U/L ≈ 10^{-5} s^{-1}
- That is to say that mid-latitude synoptic scale motions are quasi-nondivergent (horizontally)
- **The approximate vorticity equation is**:

$$\frac{d_H}{dt}(\zeta_r + f) \cong f\frac{\partial w}{\partial z} \cong -f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

The potential vorticity

- The conservation of vorticity expressed by the vorticity equation is only valid if there is no vertical movement of air (as we have neglected w)
- The vorticity is not conserved if lifting of air (i.e., $w \neq 0$) occurs
- By combining the vorticity equation with the conservation of mass, a new quantity is derived: the Potential Vorticity (PV)
- For this quantity a more general conservation law can be derived, which is directly applicable to the atmosphere
- The conservation law for the potential vorticity is valid for a barotropic atmosphere, i.e. if the isolines of pressure and temperature are parallel
- Two equivalent definitions for the potential vorticity (after Ertel and Rossby) exist

Hans Ertel (1904-1971) was appointed in 1946 Professor of Geophysics at the University Berlin, as well as director of the Institute of Meteorology and Geophysics there.

Carl-Gustaf Arvid Rossby (1898–1957) was a Swedish born American meteorologist who first explained the large-scale motions of the atmosphere in terms of fluid mechanics.

Rossby's Potential Vorticity

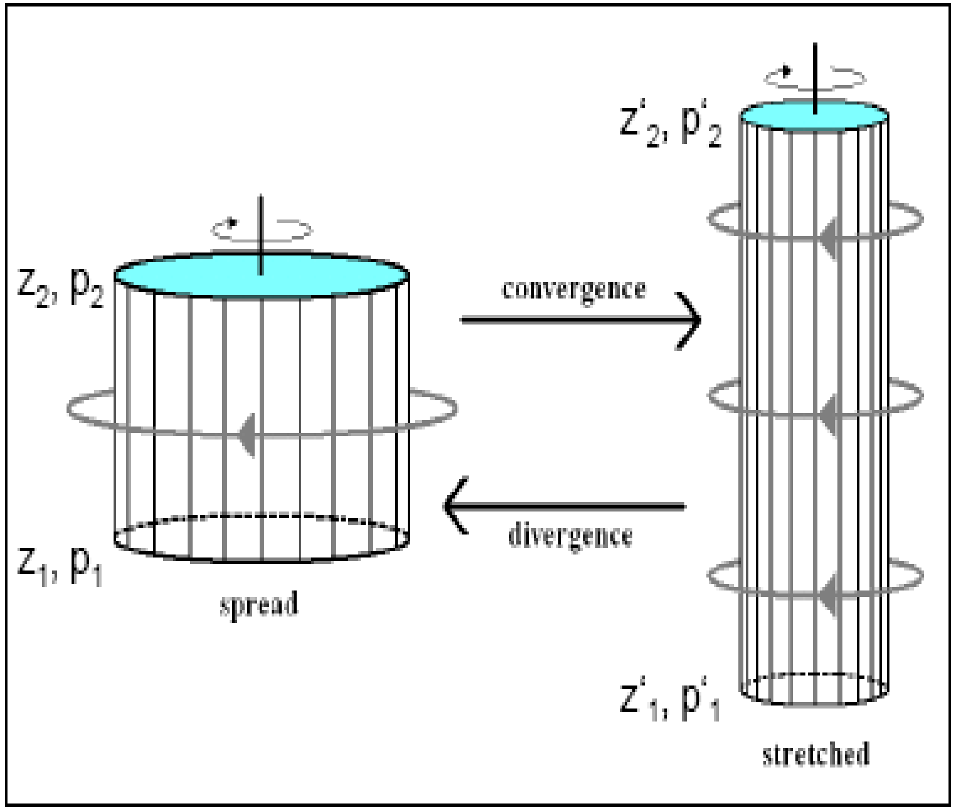
- Rossby's potential vorticity is a quantity, related to an air column of finite vertical extent $\Delta z = z_2 - z_1$
- We assume that the air parcel is bounded at its bottom and top by surfaces of defined pressure or temperature
- $\Delta p = p(z_2) - p(z_1)$ is the pressure difference between top and bottom of the air column

- Integrating the continuity equation $$\frac{1}{\rho}\frac{d\rho}{dt} + \vec{\nabla}\cdot\vec{v} = 0$$

  from z_1 to z_2 yields (after several re-arrangements):

$$\overrightarrow{\nabla_H}\vec{v} = -\frac{1}{\Delta p}\frac{d(\Delta p)}{dt}$$

- This equation relates temporal changes in vertical pressure differences to the horizontal divergence of the wind field



Rossby's Potential Vorticity

- Now we combine the continuity equation in the form $\overrightarrow{\nabla_H}\vec{v} = -\frac{1}{\Delta p}\frac{d(\Delta p)}{dt}$ [Lagrangian form with r=-Δp/(g Δz)] with the vorticity equation:

$$\frac{d\zeta_a}{dt} + \zeta_a \overrightarrow{\nabla_H}\vec{v} = 0 \quad \text{or} \quad \overrightarrow{\nabla_H}\vec{v} = -\frac{1}{\zeta_a}\frac{d\zeta_a}{dt}$$

- Giving: $$-\frac{1}{\Delta p}\frac{d(\Delta p)}{dt} + \frac{1}{\zeta_a}\frac{d\zeta_a}{dt} = 0 \quad \text{or} \quad -\frac{\zeta_a}{(\Delta p)^2}\frac{d(\Delta p)}{dt} + \frac{1}{\Delta p}\frac{d\zeta_a}{dt} = 0$$

- This is equivalent to $$\frac{d(PV_R)}{dt} = 0 \quad \text{or} \quad PV_R = \frac{\zeta_a}{\Delta p} = const$$

with PV_R (Rossby's potential vorticity) being a conserved quantity

Ertel's Potential Vorticity

- Assuming the upper and lower boundary layers of the air parcel having the potential temperatures θ_1 and θ_2, respectively, the difference $\Delta\theta$ is conserved during adiabatic ascent or descent:

$$\Delta\theta = \theta_2 - \theta_1 = (z_2 - z_1)\frac{d\theta}{dz} = const$$

- Neglecting the change in density due to different actual temperatures, we can express the altitude difference as pressure difference:

$$z_2 - z_1 = -\frac{\Delta p}{\rho g} \quad \text{thus} \quad \Delta p = const\,\frac{\rho g}{d\theta/dz}$$

- Inserting this expression of Δp in eq. for PVR we arrive to Ertel's PV:

$$PV_E = \frac{\zeta_a}{\rho}\frac{d\theta}{dz} = const \quad \xrightarrow[\mathrm{dp=-\rho g dz}]{} \quad PV_E^* = \zeta_a \frac{d\theta}{dp} = const$$

Some Properties of (Ertel‘s) PV

$$PV_E = \frac{\zeta_a}{\rho}\frac{d\theta}{dz} = const \quad \xrightarrow[\text{dp=-}\rho\text{gdz}]{} \quad PV_E^* = \zeta_a \frac{d\theta}{dp} = const$$

- Dimensions of PVE and PVE*:

$$[PV_E] = \frac{s^{-1}}{kg\, m^{-3}}\frac{K}{m} = \frac{K\, m^2}{kg\, s} \qquad [PV_E^*] = s^{-1}\frac{K}{Pa}$$

- Frequently used unit:

$$10^{-6}\frac{K\, m^2}{kg\, s} = 1\, PVU\ (PV\text{-}Unit)$$

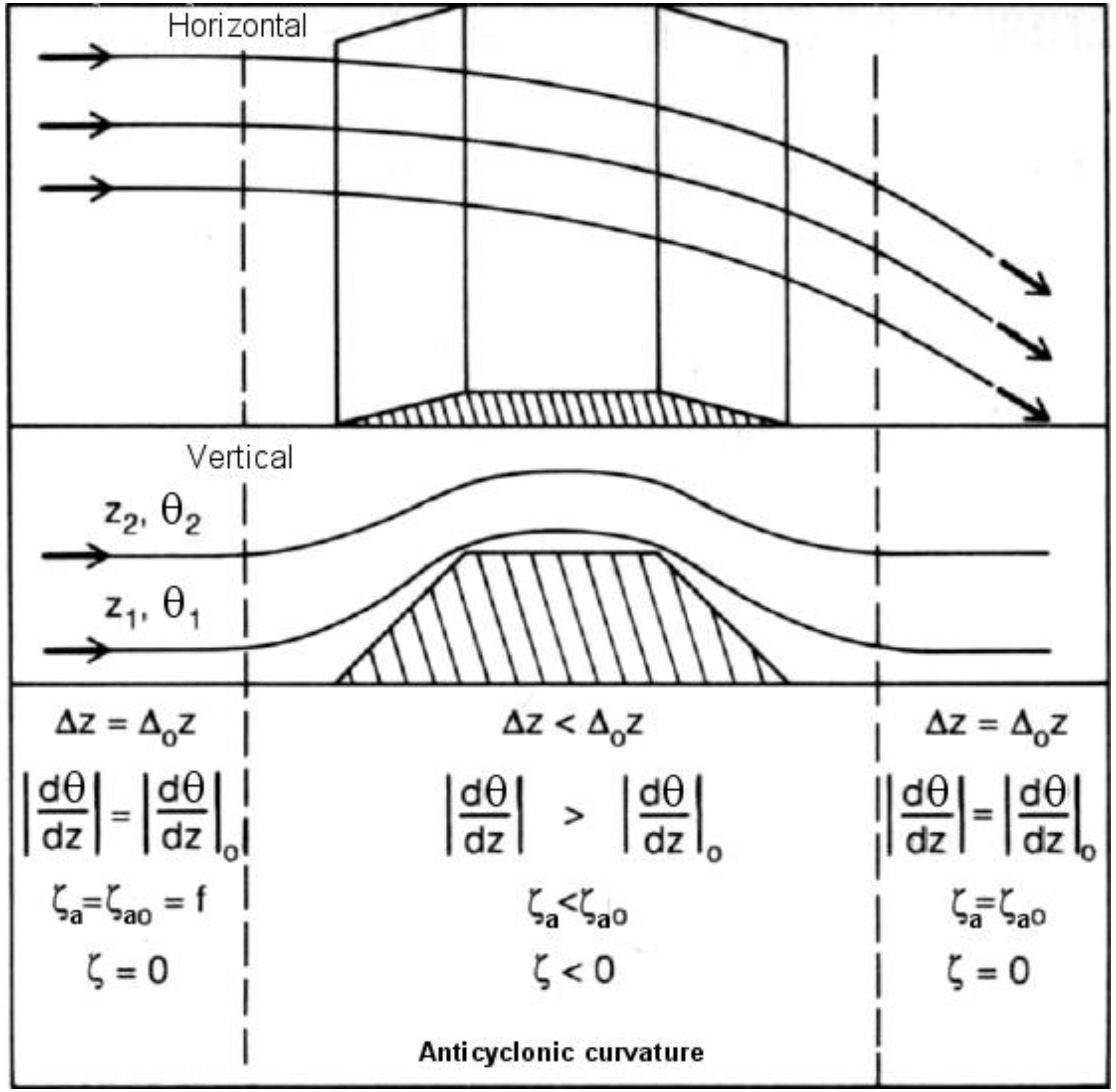
Summary

- The conservation of vorticity (curl of the wind field) expresses the conservation of angular momentum
- The absolute vorticity (sum of relative vorticity and Coriolis parameter) also considers the rotation of the Earth
- The vorticity follows a continuity equation (the vorticity equation) with the curl of the external force field as source term
- The potential vorticity (PV) is an important concept in atmospheric dynamics, as it connects the continuity equation with the vorticity equation and is also valid for vertical movements
- PV is a conserved quantity in a barotropic atmosphere

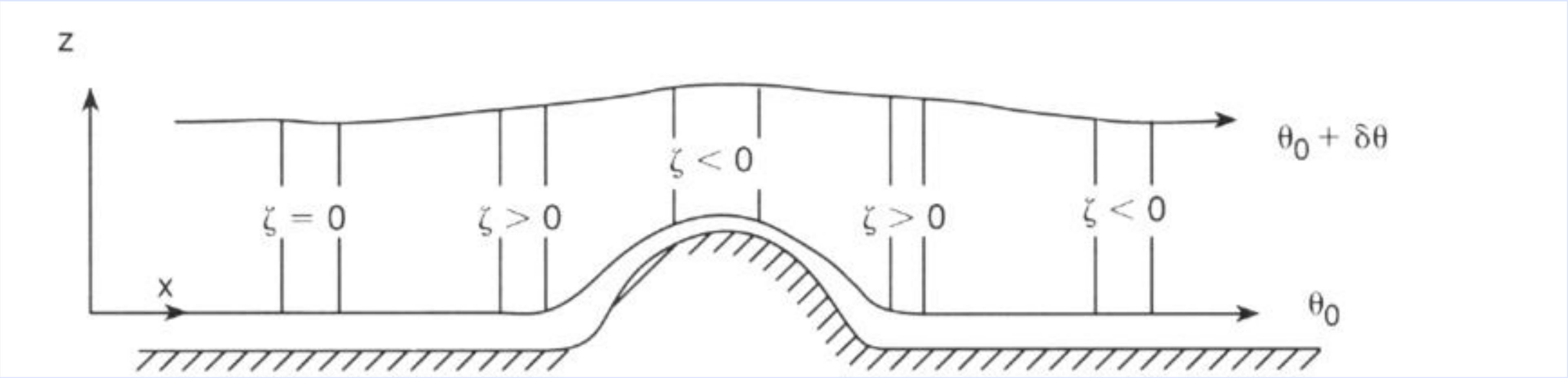
Example: Change of Wind Direction in Flows Across Obstacles

$$PV_E = \frac{\zeta_a}{\rho}\frac{d\theta}{dz} = const$$

- Change of wind direction in a flow across an obstacle (hill) due to the conservation of PV
- Here a constant Coriolis parameter is assumed
- Note that the direction of deflection is independent of the direction of flow
- Adapted from: W. Roedel, 1992, p. 107

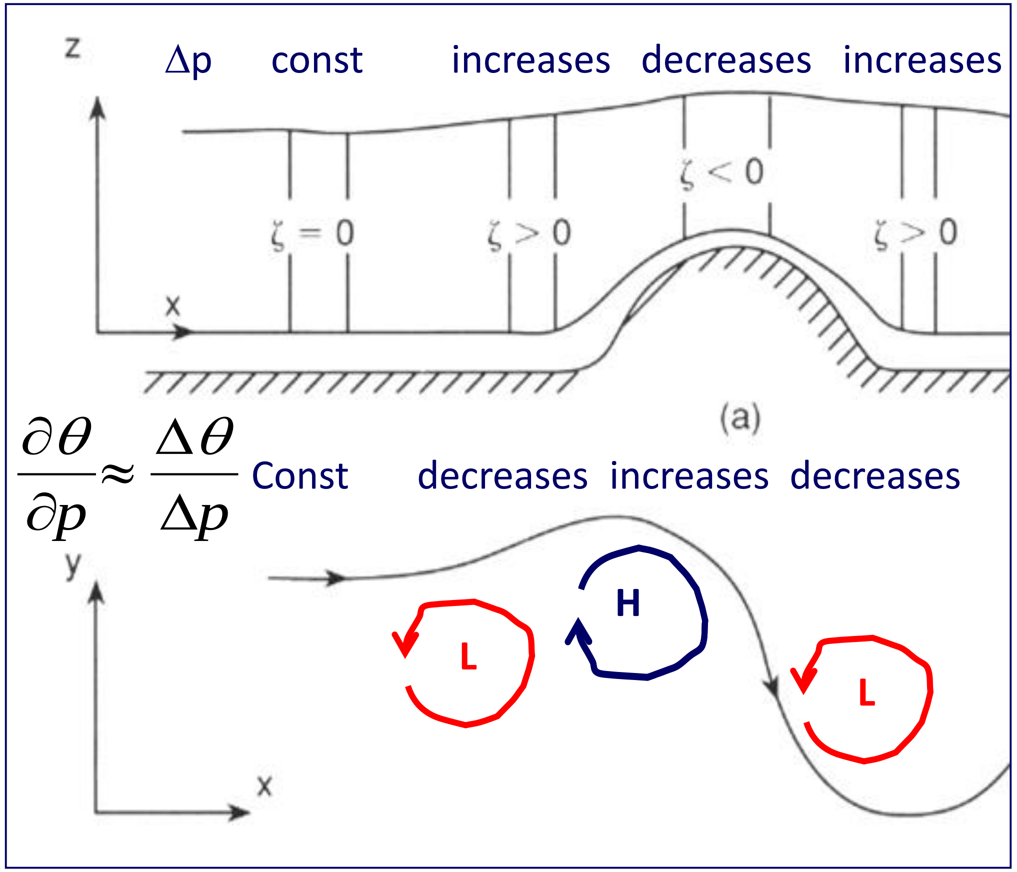


Adiabatic Flow Over a Mountain Range (Holton, 1992)



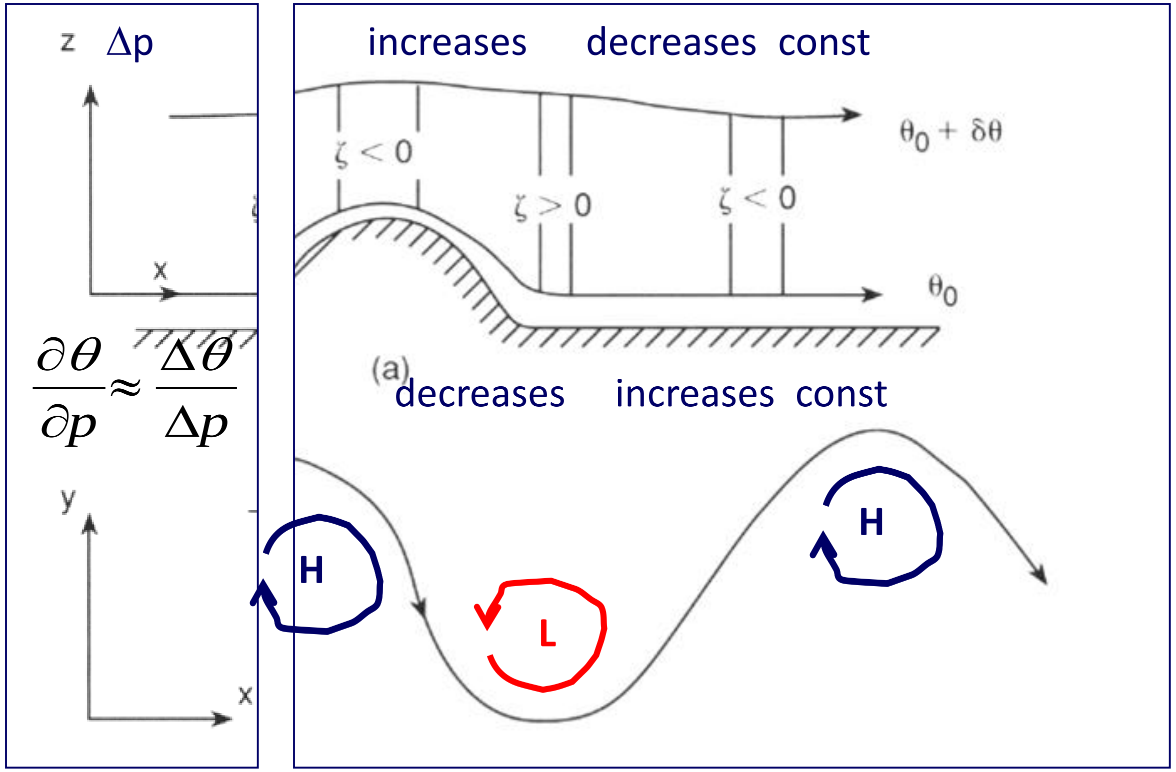
- A typical example is a zonal flow over a mountain range in West→East direction, as represented in the figure above
- Suppose that the initial relative vorticity is null upwind
- If the flow is adiabatic, an air column between the two surfaces θ_0 and $\theta_0+d\theta$ will remain confined between these surfaces during the passage above the mountain
- Lower θ surface roughly follows the contour of the mountain
- Highest one has a "draft" less pronounced that decreases with the quote and disappears at sufficiently high θ levels
- $\partial\theta/\partial p$ varies so it is possible to stretch or compress the columns of fluid, but the potential vorticity PVE will be conserved

Adiabatic Flow Over a Mountain Range (Holton, 1992)



- When the flow reaches the bottom of the mountain, $\partial\theta/\partial p$ decreases (being the isobaric surfaces horizontal, $\delta\theta$ do not changes, $|\delta p|$ increases) then ζ increases to conserve PV_E
- The particle, approaching the mountain, curve its trajectory Northwards
- This northwards motion also causes a growth of f which reduces the ζ variation necessary to conserve PV_E
- When rise starts, air column is compressed, thus $\zeta<0$ (anticyclonic curvature) → the particle moves southwards

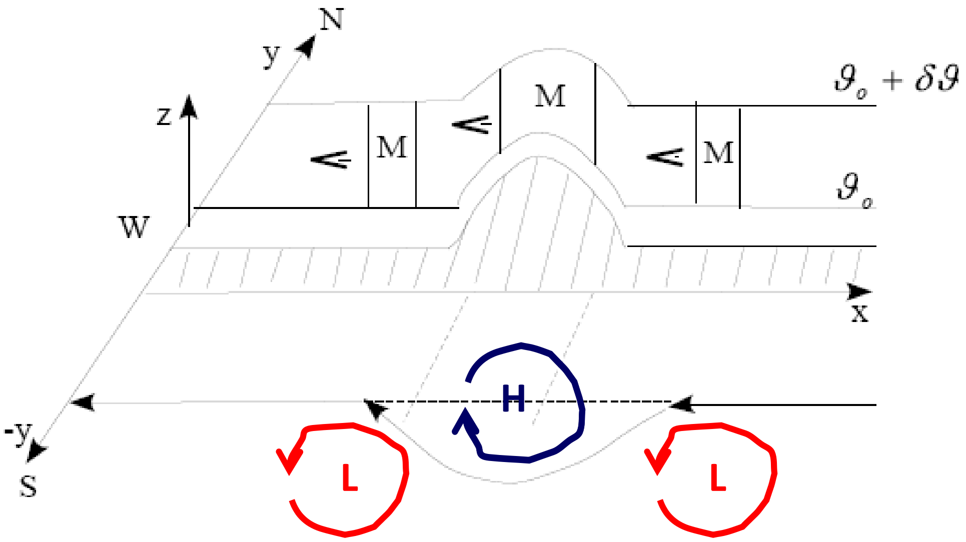
Adiabatic Flow Over a Mountain Range (Holton, 1992)



- Once crossed the mountain, the particle has a smaller f (smaller latitude) than initially, and then must rotate cyclonically (z increases) moving northwards
- Moving northwards, f increases so gradually z decreases since it becomes negative – flow will oscillate

- Note that horizontal θ-constant surfaces coincide with isobaric surfaces → in these areas the fluid is barotropic
- Near and above the mountain, the θ-constant surfaces are no longer horizontal → baroclinic atmosphere → it is essential to account for the conservation of PV_E

Adiabatic Flow Over a Mountain Range (Holton, 1992)



- When the flow is oriented East→West, the following situation occurs
- Near the bottom of the mountain, Dp increases thus ∂θ/∂p decreases and for compensation z must increase (cyclonic rotation), forcing the flow southwards

- During the mountain ascent, ∂θ/∂p increases forcing ζ to decrease (anticyclonic rotation) – when the flow is at the top of the mountain, its direction is again zonal
- During the following descent the decrease of ∂θ/∂p forces ζ to increase (cyclonic rotation), while the following weak increment of ∂θ/∂p forces ζ to decrease (anticyclonic rotation)
- Far from the mountain, the flow is again zonal and at the same latitude of the original flow, thus obscillations are absent

Summary

- The zonal western flows transform the disturbances created by an orographic relief in wave fluctuations downwind to the mountains
- The zonal eastern flows smooth the disturbances created by an orographic relief
- Thus, the answer of the two kind of flows to vertical disturbances is different from the case in which a constante thermal stratification is imposed

Rossby Waves (1)

We consider the west wind drift, as barotropic air flow with the vorticity equation:

$$\frac{\partial \zeta_a}{\partial t} = \frac{\partial \zeta}{\partial t} + \frac{\partial f}{\partial t} = \frac{\partial \zeta}{\partial t} + \frac{\partial y}{\partial t}\frac{\partial f}{\partial y} = \frac{\partial \zeta}{\partial t} + v_y \frac{\partial f}{\partial y} = 0$$

since $f = f(y)$ (only)

Further:

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial \varphi}\cdot\frac{\partial \varphi}{\partial y} = \frac{\partial}{\partial \varphi}(2\Omega \sin\varphi)\cdot\frac{1}{R} \qquad (\partial y = R\cdot\partial\varphi)$$

$$= 2\Omega\cos\varphi\cdot\frac{1}{R} \quad \Rightarrow \quad \frac{\partial \zeta}{\partial t} + v_y \underbrace{\frac{2\Omega\cos\varphi}{R}}_{\beta} = 0$$

or

$$\frac{\partial \zeta}{\partial t} - v_y \beta = 0$$

Rossby Waves (2)

$$\boxed{\frac{\partial \zeta}{\partial t} - v_y \beta = 0}$$

Initially $\zeta=0$, $v_y=0$

Some distortion of the flow changes $v_y<0$ (southward flow)

→ $\zeta>0$ = flow curves to the left

→ Eventually there will be northward flow with $v_y>0$

→ $\zeta<0$ = flow curves to the right

…

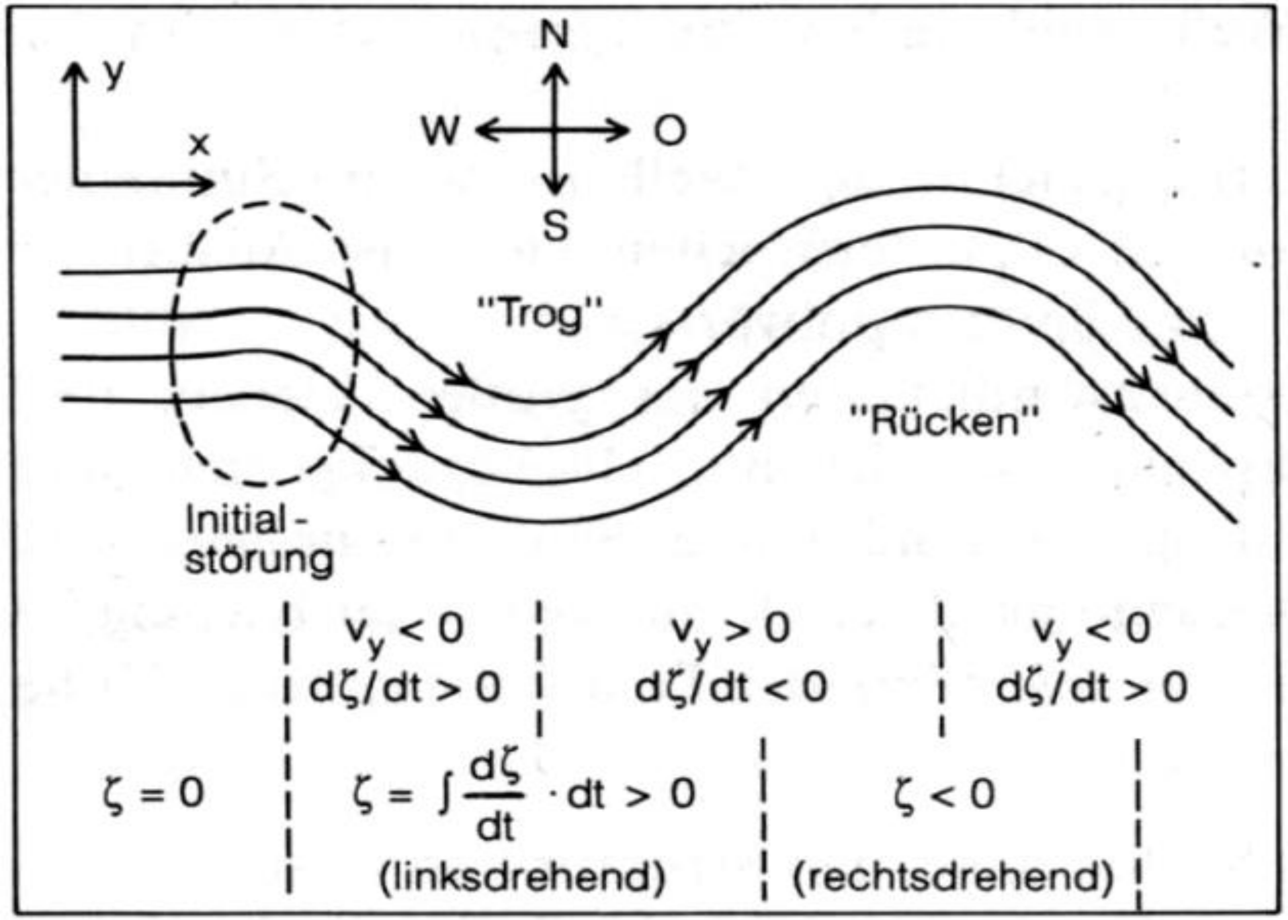
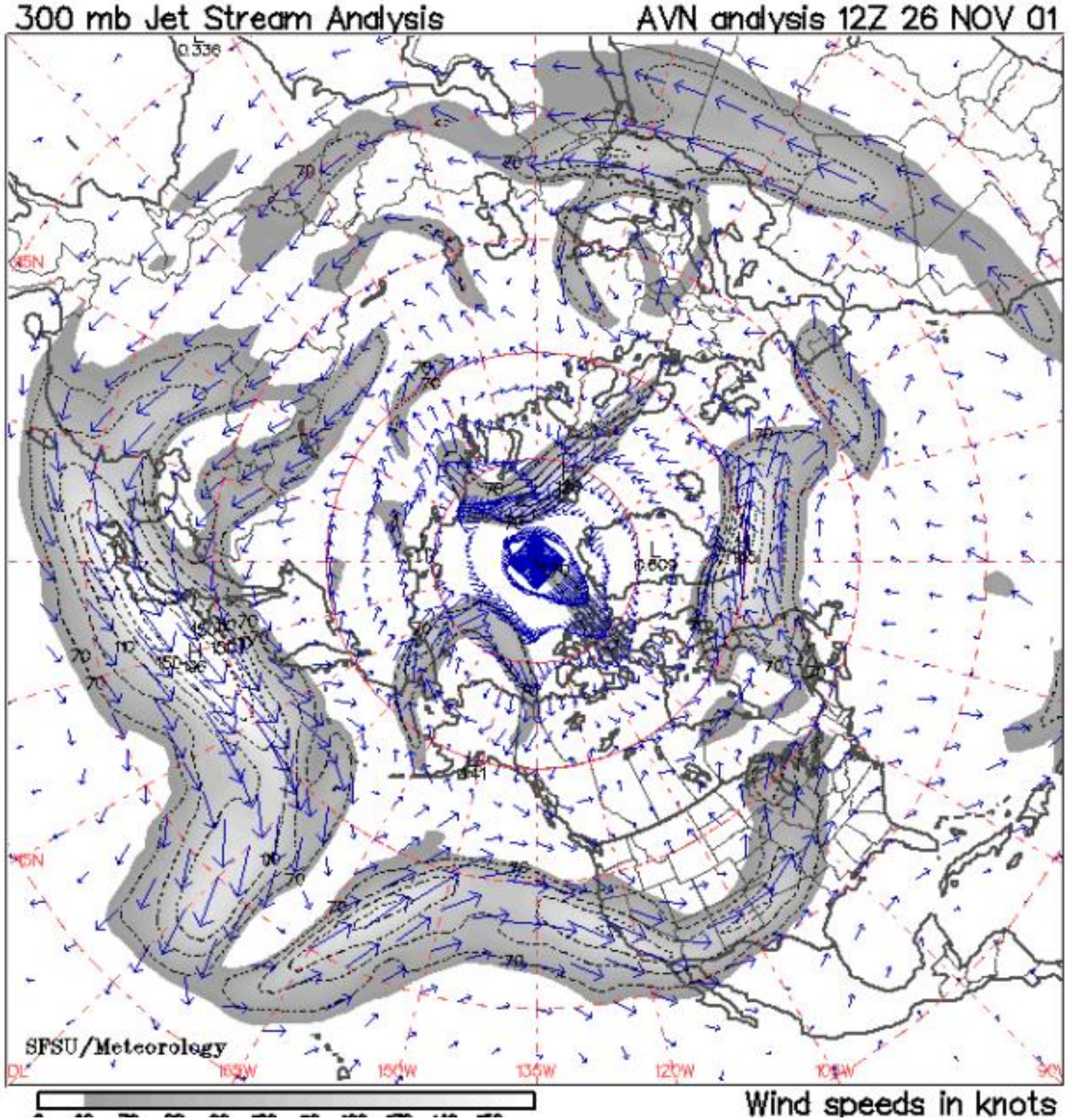


Fig. 4.5 from W. Roedel, 2000, p. 134

The Evolution of Rossby Waves

The Jet-Stream

Position of the Jet-Stream, Analyssys by San Francisco State University (http://squall.sfsu.edu/crws/jetstream.html)



Universität Heidelberg

PV and Potential Temperature Gradients

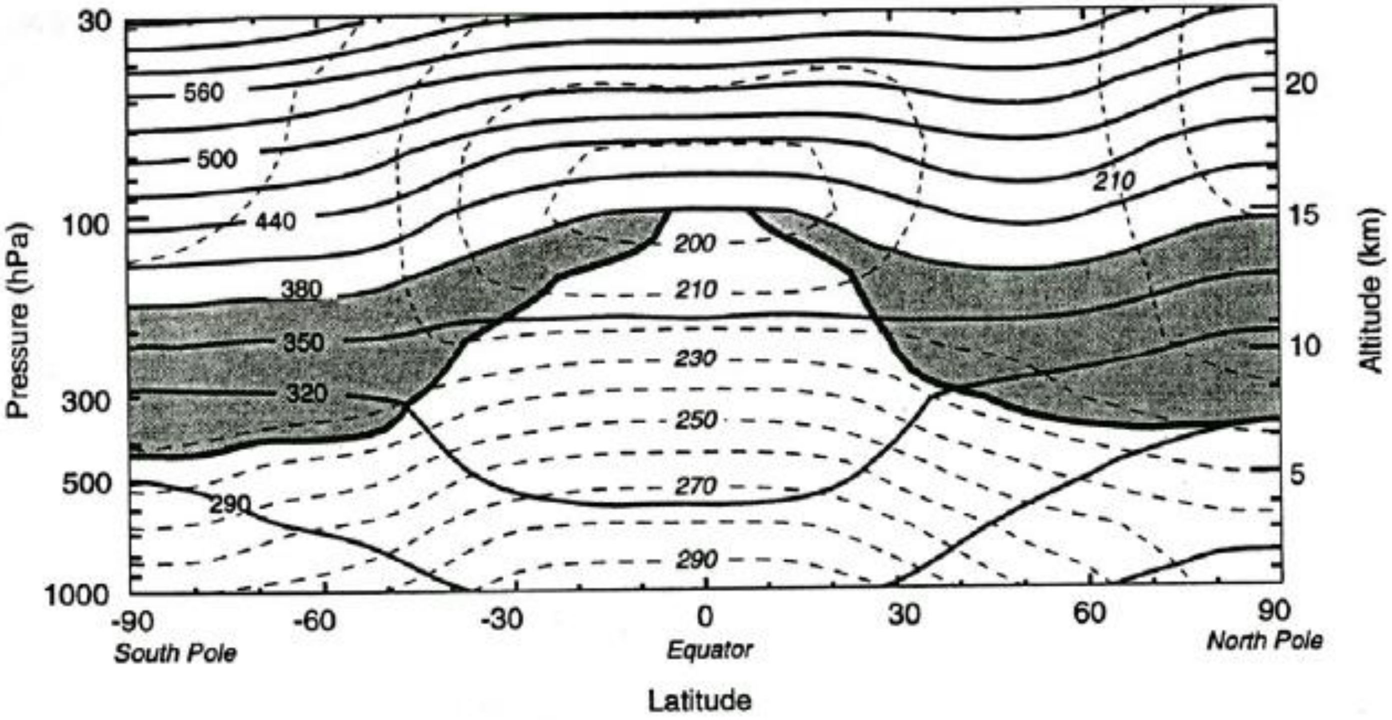


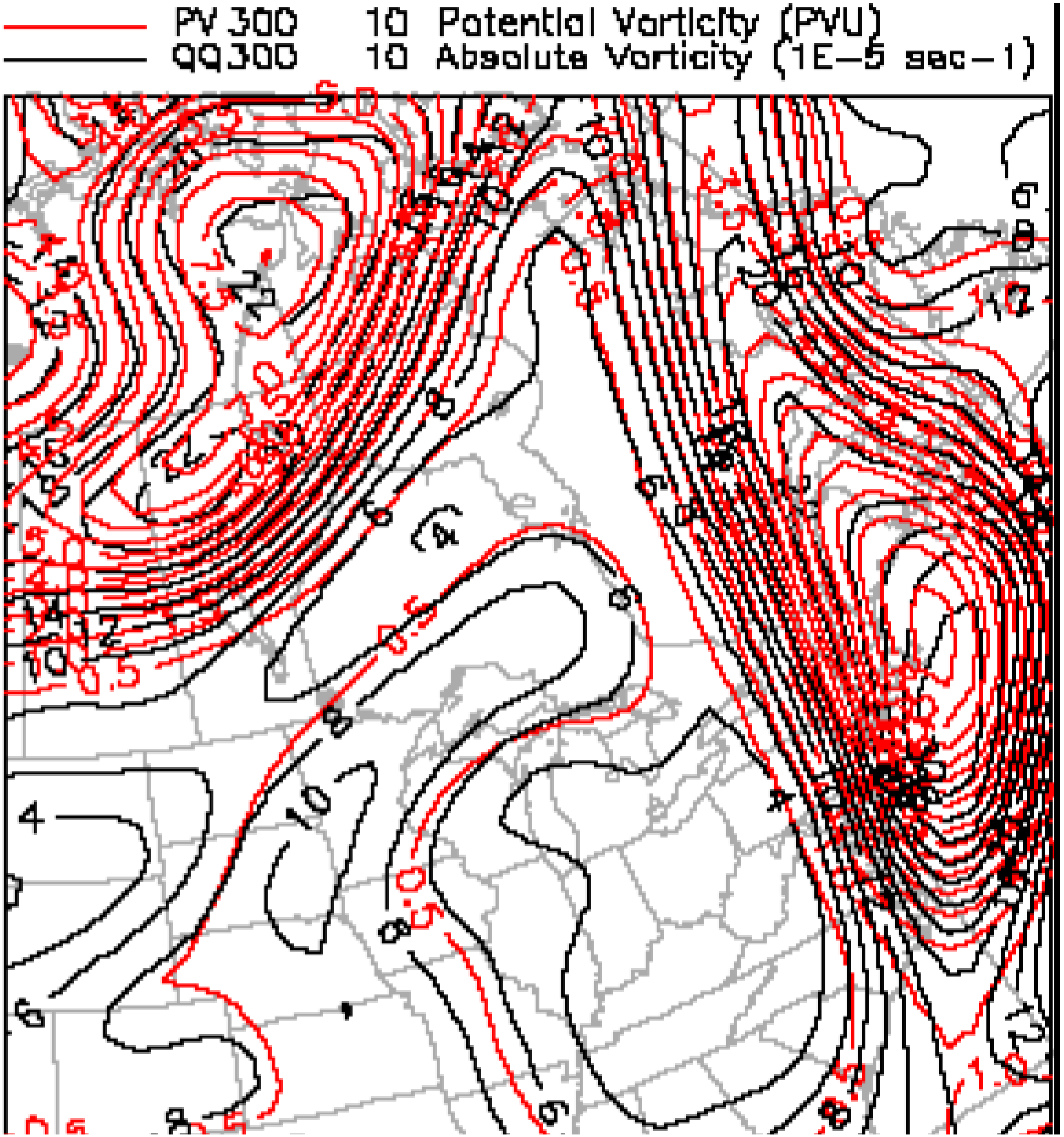
Figure 1. Latitude-altitude cross section for January 1993 showing longitudinally averaged potential temperature (solid contours) and temperature (dashed contours). The heavy solid contour (cut off at the 380-K isentrope) denotes the 2-PVU potential vorticity contour, which approximates the tropopause outside the tropics. Shaded areas denote the "lowermost stratosphere," whose isentropic or potential temperature surfaces span the tropopause. Data are from UKMO analyses [*Swinbank and O'Neill*, 1994]. In these data the tropical tropopause occurs near θ ≈ 380 K, which is somewhat higher than the mean tropical tropopause potential temperature (θ ≈ 370 K) determined from radiosonde data by *Reid and Gage* [1981]. Polar tropopauses are far more variable and less well defined than might be suggested here. Figure courtesy of C. Appenzeller.

Stratosphere, $d\Theta/dz \approx 20$ K/km

Troposphere, $d\Theta/dz \approx 5$ K/km

Stratosphere: $d\Theta/dz \approx$ higher, density lower than troposphere

Comparison of Absolute and Potential Vorticity at an Altitude of ca. 8 km.



ETA-32 Map for
Nov 6, 2001, 10:00

http://meteocentre.com/