Summary of governing equations

Textbooks and web sites references for this lecture:

- James R. Holton, An Introduction to Dynamic Meteorology, Academic Press, 1992, ISBN 0-12-354355-X (§ 1-3)

Momentum equation (Navier-Stokes)

- The momentum equation can be written assuming that the only real forces acting on the atmosphere are pressure gradient force, gravitation and friction, and the apparent Coriolis force is written explicitly while the apparent centrifugal force has been included in the gravity term **g**:

$$\frac{d\vec{V}}{dt} = -\frac{1}{\rho}\nabla p - 2\vec{\Omega}\times\vec{V} - g\hat{k} + \overrightarrow{F_r}$$

- This equation can be splitted into the three components:

$$\frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v\sin\phi - 2\Omega w\cos\phi \quad + F_{rx}$$

$$\frac{dv}{dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad - 2\Omega u\sin\phi \quad + F_{ry}$$

$$\frac{dw}{dt} - \frac{u^2 + v^2}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial z} \quad + 2\Omega u\cos\phi + g + F_{rz}$$

Synoptic scale analysis

- Scaling is a technique useful to determine whether some terms in the equation are negligible for motions of meteorological concern
- Elimination of terms on scaling considerations simplify mathematics and allows to eliminate and filter unwanted types of motions, like sound waves
- The following scales are assumed characteristics for mid-latitude synoptic systems

U	$\approx$	$10\ m\ s^{-1}$	wind speed scale
W	$\approx$	$0.01\ m\ s^{-1}$	vertical velocity scale
L	$\approx$	$10^6\ m$	horizontal length scale
H	$\approx$	$10^4\ m$	vertical length scale
p	$\approx$	$1000\ hPa\ (10^5\ Pa)$	mean surface pressure
Δp	$\approx$	$10\ hPa\ (10^3\ Pa)$	pressure horizontal variation scale
$\Delta\rho/\rho$	$\approx$	10^{-2}	fractional density fluctuation
$T \approx L/U$	$\approx$	$10^5\ s$	time scale
ν	$\approx$	$10^{-1}\ m^2\ s^{-1}$	kinematic air diffusivity
f_0	$\approx$	$10^{-4}\ s^{-1}$	Coriolis parameter
a	$\approx$	$10^7\ m$	Earth radius

Scale analysis of horizontal equation

- It is possible to scale each term of each horizontal equation using previously defined scales and the definition of the Coriolis parameter:
- $f_0 = 2\ \Omega \sin f_0 = 2\ \Omega \cos \varphi_0 = 10^{-4}\ s^{-1}$

$$x-eq. \quad \frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v\sin\phi - 2\Omega w\cos\phi + F_{rx}$$

$$y-eq. \quad \frac{dv}{dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin\phi \qquad + F_{ry}$$

$Scales$	U^2/L	$\frac{U^2}{a}$	$\frac{UW}{a}$	$\frac{\delta P}{\rho L}$	$f_0 U$	$f_0 W$	$\frac{\nu U}{L^2}$
(ms^{-2})	10^{-4}	10^{-5}	10^{-8}	10^{-3}	10^{-3}	10^{-6}	10^{-12}

Geostrophic wind

- Retaining only two bigger terms from the analysis equation, it is possible to write the first approximation geostrophic balance:

$$-fv \approx -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad fu \approx -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\mathbf{V}_{\mathrm{g}} \equiv \mathbf{k} \times \frac{1}{\rho f} \nabla p$$

- This equation is diagnostic (no reference to time) and allows the definition of geostrophic wind as balance between Coriolis and pressure gradient forces

Approximate prognostic equation

- To obtain prediction equation it is necessary to retain also acceleration term. The resulting approximate horizontal equations can be written as:

$$\frac{du}{dt} = fv - \frac{1}{\rho}\frac{\partial p}{\partial x} = f\left(v - v_g\right)$$

$$\frac{dv}{dt} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y} = -f\left(u - u_g\right)$$

- In these equations, acceleration term (small) is given by the difference between two larger terms → large errors in its determination
- Measure of magnitude of acceleration compared with Coriolis force is given by the ratio of their characteristic scales, called Rossby number:

$$Ro \equiv \frac{U^2 / L}{f_0 U} = \frac{U}{f_0 L}$$

Scale analysis of vertical equation

- By scaling in the same way vertical equation, we arrive to:

$$
\begin{array}{lcccccccc}
z-eq. & \dfrac{dw}{dt} & -\dfrac{u^2+v^2}{a} & = & -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} & +2\Omega u\cos\phi & +g & +F_{rz} \\
Scales & UW/L & \dfrac{U^2}{a} & & \dfrac{\Delta p}{\rho H} & f_0 U & g & \dfrac{\nu W}{H^2} \\
(ms^{-2}) & 10^{-7} & 10^{-5} & & 10 & 10^{-3} & 10 & 10^{-15}
\end{array}
$$

- The biggest terms give the hydrostatic approximation (valid also for perturbations):

$$\frac{1}{\rho_0}\frac{dp_0}{dz} \equiv -g \qquad\qquad \frac{1}{\rho}\frac{dp'}{dz} = -g$$

Continuity equation

- It can be written in one of the two forms:

$$\frac{1}{\rho}\frac{d\rho}{dt}+\nabla\circ\vec{V}=0 \qquad\qquad \frac{\partial\rho}{\partial t}+\nabla\circ\left(\rho\vec{V}\right)=0$$

- Scale analysis of the second show that local time derivative of pressure is an order of magnitude lower, leaving:

$$\nabla\circ\left(\rho\vec{V}\right)=0$$

- Notice that this is different from the incompressible fluid hypothesis, valid in the boundary layer

$$\nabla\circ\vec{V}=0$$

Thermodynamic energy equation

- Starting from the usual form of thermodynamic equation:

$$\delta Q = \delta L + dU = pdV + C_v dT \quad \text{Per unit mass} \quad \delta q = Jdt = pd\alpha + c_v dT$$

where J=dq/dt is the rate of heating per unit mass owing to radiation, conduction, and latent heat release, and α=1/r, we can use continuity equation to show that:

$$\frac{1}{\rho}\frac{d\rho}{dt} + \nabla \circ \vec{V} = 0 \Rightarrow \frac{1}{\rho^2}\frac{d\rho}{dt} + \frac{1}{\rho}\nabla \cdot \vec{V} = 0 \Rightarrow \frac{1}{\rho}\nabla \circ \vec{V} = -\frac{1}{\rho^2}\frac{d\rho}{dt} = \frac{d\alpha}{dt}$$

- For dry air, internal energy per unit mass is defined as $e=c_v T$
- This allows to rewrite thermodynamic equation as:

$$\delta q = Jdt = pd\alpha + c_v dT \Rightarrow J = \frac{de}{dt} - \frac{p}{\rho^2}\frac{d\rho}{dt} = \frac{de}{dt} + \frac{p}{\rho}\nabla \circ \overline{V}$$

$$\rho\frac{de}{dt} = -p\nabla \circ \vec{V} + \rho J$$

Thermodynamic energy equation

$$\rho \frac{de}{dt} = -p\nabla \circ \vec{V} + \rho J$$

- Let's suppose no convergence/divergence ($\nabla \cdot \vec{V} = 0$): in this case, a positive (negative) J produces positive (negative) variation of e, e.g. positive (negative) variation of T

- Let's suppose no heating rate ($J = 0$): in this case, a positive (negative) convergence ($\nabla \cdot \vec{V} > 0$) produces negative (positive) variation of e, e.g. negative (positive) variation of T