Physics of the Atmosphere

# Lecture 12: Conservation Laws in the Atmosphere (III)

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# Advanced Topics (Material from 2019 Course)

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# Conservation of Energy

- **Thermodynamic Equation:** $c_p \frac{dT}{dt} - \alpha \frac{dp}{dt} = \dot{Q}$

$$\frac{\dot{Q}}{T} = \frac{c_p}{T}\frac{dT}{dt} - \frac{\alpha}{T}\frac{dp}{dt} = c_p \frac{d\ln T}{dt} - \frac{R_d}{p}\frac{dp}{dt} = c_p \frac{d\ln T}{dt} - R_d \frac{d\ln p}{dt}$$

  - Introducing potential temperature and differentiating it:

$$\theta = T\left(\frac{p_0}{p}\right)^{R_d/c_p} \Rightarrow \frac{d\theta}{dt} = \frac{dT}{dt}\frac{\theta}{T} - \frac{R_d}{c_p}\frac{\theta}{p}\frac{dp}{dt} \Rightarrow c_p \frac{d\ln\theta}{dt} = c_p \frac{d\ln T}{dt} - R_d \frac{d\ln p}{dt} = \frac{\dot{Q}}{T} = \frac{ds}{dt}$$

  - As $\dot{Q}/T$ represents the variation of entropy $s$, the thermal energy equation analyzed the variation of entropy along the motion.
  - When $s$ is constant, also the potential temperature conserves.



# Conservation of Energy

- **Thermodynamic Equation:** $c_p \frac{dT}{dt} - \alpha \frac{dp}{dt} = \dot{Q}$

$$c_p \frac{d\ln\theta}{dt} = c_p \frac{d\ln T}{dt} - R_d \frac{d\ln p}{dt} = \frac{\dot{Q}}{T} = \frac{ds}{dt}$$

  - For adiabatic reversible processes in dry air, the variations of potential temperature are proportional to entropy variations.
  - If a particle conserves entropy along its trajectory, it conserves also potential temperature.
  - $\theta(x, y, z, t) = \theta_0(z) + \theta'(x, y, z, t)$

$$c_p \frac{d\ln\theta}{dt} = \frac{\dot{Q}}{T} = \frac{ds}{dt} \Rightarrow \frac{1}{\theta}\frac{d\theta}{dt} = \frac{\dot{Q}}{c_p T} \Rightarrow \frac{1}{\theta_0}\left(\frac{\partial\theta'}{\partial t} + v \bullet \nabla_H \theta'\right) + w\frac{\partial \ln\theta_0}{\partial z} = \frac{\dot{Q}}{c_p T}$$

# Conservation of Energy

- **Thermodynamic Equation:** $c_p \frac{dT}{dt} - \alpha \frac{dp}{dt} = \dot{Q}$
  - The diabatic heating rate of the atmosphere $d\dot{Q}/dt$ is mainly produced by net radiation, and occasionally (in the clouds) by latent heat flux.
  - In the troposphere net radiation warming can produce a diabatic trend in temperature equal to 1°C/day.
  - Horizontal temperature fluctuations at synoptic scale are of the order of 4°C.
  - Thus, the two processes able to induce variations (both of 4°C/day) in the air potential temperature are
    - Horizontal thermal advection
    - Vertical thermal convection

$$\left(\frac{\dot{Q}}{c_p}\right)_{hor.adv.} = \frac{T}{\theta_0}\left(\frac{\partial \theta'}{\partial t} + \vec{u}\cdot\overrightarrow{\nabla\theta'}\right) \cong \frac{\theta' U}{L} \approx +4°\text{C day}^{-1}$$

$$\left(-\frac{\dot{Q}}{c_p}\right)_{ver.con.} = w\left(\frac{T}{\theta_0}\frac{\partial \theta_0}{\partial z}\right) = w(\gamma_a - \gamma) \approx -4°\text{C day}^{-1}$$

$\gamma_a - \gamma \approx 4$°C/km (stable stratification)



# Conservation of Energy

- **Thermodynamic Equation:** $c_p \frac{dT}{dt} - \alpha \frac{dp}{dt} = \dot{Q}$
  - Thus thermal energy equation, in absence of diabatic heating, can be simplified to:

$$\frac{\partial \theta'}{\partial t} + v \bullet \nabla_H \theta' + w\frac{\partial \theta_0}{\partial z} = 0$$

  - This equation links the horizontal advection (>0) with the vertical convection (<0) and explains how these two processes nearly compensate

$$\left(\frac{d\theta'}{dt}\right)_{h.a.} + \left(w\frac{\partial \theta_0}{\partial z}\right)_{v.c.} = 0$$

# Conservation of Energy

- **An alternative way to express the same law**

Power conservation

$$\frac{D}{Dt}\left[\rho\left(\underset{\text{internal energy}}{E}+\underset{\text{kinetic energy}}{\frac{1}{2}\mathbf{v}\bullet\mathbf{v}}\right)dV\right]=\frac{D}{Dt}\left[\underset{\text{heat given to system}}{\delta Q}-\underset{\text{work done to environment}}{\delta W}\right]$$

- This law says that the variation of the total thermodynamic energy is equal to the difference among the heat given to the fluid element and the work done by the system itself (in the unit of time) against external environment forces.
- The external forces acting on the Lagrangian fluid element of volume $\delta V$ are: surface (pressure gradient and friction) and volume (gravitation and Coriolis) forces.
- The power produced by these forces (which meteorologist prefer to call "work per unit of time") is given by their scalar product with velocity.
- The power, and not the work, is considered, as the system evolves in time.

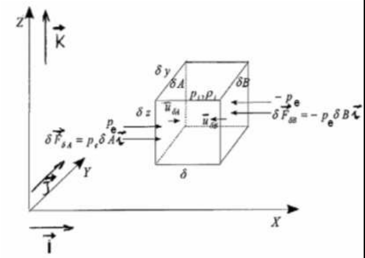
| | | | |
|---|---|---|---|
| Work ($W$) | $ML^2T^{-2}$ | kg m$^2$ s$^{-2}$; J | $W = F \cdot l$ |
| Energy ($E$) | $ML^2T^{-2}$ | kg m$^2$ s$^{-2}$; J | |
| Power ($P$) | $ML^2T^{-3}$ | kg m$^2$ s$^{-3}$; J s$^{-1}$ | $P = dW/dt$; $dE/dt$ |



# Conservation of Energy

- **Energy Equation: PGF**

$$\frac{dW_A}{dt}=p_e\delta y\delta z\frac{\delta x}{dt}=p_e u_A\delta y\delta z;\quad \frac{dW_B}{dt}=-p_e u_B\delta y\delta z$$



- The total net power developed by the pressure gradient force along $x$ axis is

$$\delta P_x = p_e\left(u_A-u_B\right)\delta y\delta z$$

- Developing $pu_B$ in series

$$(pu)_B=(pu)_A+\left[\frac{\partial}{\partial x}(pu)\right]_A\delta x+\ldots\quad\Rightarrow\left[(pu)_B-(pu)_A\right]\delta y\delta z=\delta P_x=-\left[\frac{\partial}{\partial x}(pu)\right]_A\delta x\delta y\delta z=-\left[\frac{\partial}{\partial x}(pu)\right]_A\delta V$$

$$\delta P_p=-\nabla\cdot\left(p\vec{u}\right)\delta V$$

# Conservation of Energy

- **Energy Equation: GF & CF**
  - The power developed by gravitation is:

$$\delta P_g = \rho \mathbf{g} \bullet \mathbf{v} \delta V = -\rho g w \delta V$$

  - The power developed by the Coriolis force is zero by definition, as Coriolis force is perpendicular to velocity and thus cannot produce work
  - The thermodynamic equations is thus

$$\frac{D}{Dt}\left[\rho\left(E+\frac{1}{2}\mathbf{v}\bullet\mathbf{v}\right)\delta V\right] = \rho\delta V\frac{D}{Dt}\left(E+\frac{1}{2}\mathbf{v}\bullet\mathbf{v}\right)+\left(E+\frac{1}{2}\mathbf{v}\bullet\mathbf{v}\right)\overbrace{\frac{D(\rho\delta V)}{Dt}}^{0} = -\nabla\bullet(p\mathbf{v})\delta V - \rho g\hat{\mathbf{k}}\bullet\mathbf{v}\delta V + \rho\dot{Q}\delta V$$

    - Here $Q$ indicates every source or sink of thermal energy (radiant energy, latent or sensible heat flux)
    - $\dot{Q}$ indicates the time variation of $Q$



# Conservation of Energy

- **Conservation of Thermal Energy:**

$$\frac{D}{Dt}\left[\rho\left(E+\frac{1}{2}\mathbf{v}\bullet\mathbf{v}\right)\delta V\right] = \rho\delta V\frac{DE}{Dt}+\rho\delta V\frac{D\mathbf{v}}{Dt}\bullet\mathbf{v} = -\mathbf{v}\bullet\nabla p\delta V - p\nabla\bullet\mathbf{v}\delta V - \rho g w\delta V + \rho\dot{Q}\delta V$$

  - Dividing by $\delta V$: $\rho\frac{DE}{Dt}+\rho\frac{D\mathbf{v}}{Dt}\bullet\mathbf{v} = -\mathbf{v}\bullet\nabla p - p\nabla\bullet\mathbf{v} - \rho g w + \rho\dot{Q}$
  - Momentum equ. neglecting $\mathbf{a}_{CF}$ and $\mathbf{a}_{FR}$:

Equation of mechanic energy (per unit time)

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla p - \cancel{2\boldsymbol{\Omega}\times\mathbf{v}} - g\hat{\mathbf{k}} + \cancel{\boldsymbol{\tau}} \quad \Rightarrow \quad \rho\frac{D\mathbf{v}}{Dt}\bullet\mathbf{v} = -\mathbf{v}\bullet\nabla p - \rho g\hat{\mathbf{k}}\bullet\mathbf{v}$$

  - Equation of thermal (internal) energy (per unit time):

$$\rho\frac{DE}{Dt} = -p\nabla\bullet\mathbf{v} + \rho\dot{Q}$$

# Conservation of Energy

- **Conservation of Mechanical Energy:**

$$\rho \frac{D\mathbf{v}}{Dt} \bullet \mathbf{v} = -\mathbf{v} \bullet \nabla p - \rho g \hat{\mathbf{k}} \bullet \mathbf{v} = -\mathbf{v} \bullet \nabla p - \rho g w$$

  - Recalling the geopotential: $gw = g\frac{dz}{dt} = \frac{d\Phi}{dt}$
  - Then $$\rho \frac{D\mathbf{v}}{Dt} \bullet \mathbf{v} + \rho g w = \rho \frac{D}{Dt}\left(\underbrace{\frac{1}{2}\mathbf{v} \bullet \mathbf{v}}_{\text{kinetic energy}} + \underbrace{\Phi}_{\text{potential energy}}\right) = -\mathbf{v} \bullet \nabla p$$
  - This equation shows that the variation, in the unit of time, following the motion of the fluid, of the sum of kinetic and gravitational potential energy (i.e. mechanical energy) equals the work done by pressure gradient forces, in the unit of time



# Conservation of Mass

- **Continuity Equation: Scale Analysis**

| | |
|---|---|
| $U \sim 10\ \mathrm{m\ s^{-1}}$ | horizontal velocity scale |
| $W \sim 1\ \mathrm{cm\ s^{-1}}$ | vertical velocity scale |
| $L \sim 10^6\ \mathrm{m}$ | length scale [$\sim 1/(2\pi)$ wavelength] |
| $H \sim 10^4\ \mathrm{m}$ | depth scale |
| $\delta P/\rho \sim 10^3\ \mathrm{m^2\ s^{-2}}$ | horizontal pressure fluctuation scale |
| $L/U \sim 10^5\ \mathrm{s}$ | time scale |

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \bullet \mathbf{v} = 0$$

- $\rho(x, y, z, t) = \rho_0(z) + \rho'(x, y, z, t)$

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \bullet \mathbf{v} = \frac{1}{\rho_0}\left(\frac{\partial \rho_0}{\partial t} + \frac{\partial \rho'}{\partial t} + \mathbf{v} \bullet \nabla_H \rho_0 + \mathbf{v} \bullet \nabla_H \rho' + w\frac{\partial \rho_0}{\partial z} + w\frac{\partial \rho'}{\partial z}\right) + \nabla \bullet \mathbf{v} = 0$$

$$\frac{1}{\rho_0}\left(\frac{\partial \rho'}{\partial t} + \mathbf{v} \bullet \nabla_H \rho' + w\frac{\partial \rho_0}{\partial z} + \right) + \nabla \bullet \mathbf{v} = 0$$

$$\frac{1}{\rho_0}\left(\frac{\partial \rho'}{\partial t} + \mathbf{v} \bullet \nabla_H \rho'\right) \approx \underbrace{\frac{\rho'}{\rho_0}}_{\rho'/\rho \approx 10^{-2}}\frac{U}{L} \cong 10^{-7}\,\mathrm{s}^{-1} \qquad \frac{w}{\rho_0}\frac{\partial \rho_0}{\partial z} = w\frac{\partial \ln \rho_0}{\partial z} \approx \frac{W}{H} \cong 10^{-6}\,\mathrm{s}^{-1}$$

# Conservation of Mass

- **Continuity Equation: Scale Analysis**
  - Finally as horizontal derivatives in wind divergence have same order of magnitude ($\frac{U}{L} \cong 10^{-5}\mathrm{s}^{-1}$) but opposite sign, their sum is $\propto 10^{-1}\frac{U}{L} \cong 10^{-6}\mathrm{s}^{-1}$
  - Also vertical derivative in wind divergence is scaled as $\frac{W}{H} \cong 10^{-6}\mathrm{s}^{-1}$
  - The term $\propto 10^{-7}$ can be neglected, giving:

$$\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+w\frac{\partial \ln \rho_0}{\partial z} \approx \frac{1}{\rho_0}\nabla \bullet (\rho_0 \mathbf{v}) = 0$$

  - The mass flow, evaluated using the density of base state $\rho_0(z)$ averaged on the horizontal plane, is non-divergent (Eulerian, or local, non compressibility)
  - This condition must not be confused with the true (Lagrangian, or substantial) non-compressibility, given by $D\rho/Dt = 0$, which means divergence zero.
  - However, if vertical motions can be neglected ($w{\sim}0$, true at synoptic scale), in the horizontal motions there is an approximate compressibility condition

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# Conservation of Mass

| | | |
|---|---|---|
| $u$ | $\frac{dx}{dt}$ | m s$^{-1}$ |
| $v$ | $\frac{dy}{dt}$ | m s$^{-1}$ |
| $w(\omega)$ | $\frac{dz}{dt}\left(\frac{dp}{dt}\right)$ | m s$^{-1}$ (Pa s$^{-1}$) |

- **Continuity Equation in Isobaric Coord.**
  - Using the hydrostatic approximation

$$\omega\frac{\partial}{\partial p} \to \frac{dp}{dt}\frac{\partial}{\partial p} \to (-\rho g)\frac{dz}{dt}\left(\frac{1}{-\rho g}\right)\frac{\partial}{\partial z} \to \frac{dz}{dt}\frac{\partial}{\partial z} \to w\frac{\partial}{\partial z}$$

  - The variable $\omega$ is called baric velocity, or also omega vertical velocity (it is measured in $\mathrm{hPa/s}$), and represents the total variation of the pressure following the motion of the fluid element.
  - The continuity equation in isobaric coordinates is much simpler and there is no explicit dependence on density

$$\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)_p + \frac{\partial \omega}{\partial p} = 0$$

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# Conservation of Energy

- **Energy Equation in Isobaric Coord.**
  - Expanding the total derivative

$$c_p \frac{dT}{dt} - \alpha \frac{dp}{dt} = \dot{Q} \;\Rightarrow\; c_p \left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + \omega \frac{\partial T}{\partial p} \right) - \frac{1}{\rho} \frac{dp}{dt} = \dot{Q}$$

  - Deriving Poisson equation (i.e. definition of $\theta$) with respect to $p$:

$$\theta = T \left( \frac{p_0}{p} \right)^{R_d / c_p} \;\Rightarrow\; \frac{d\theta}{dp} = \frac{dT}{dp} \frac{\theta}{T} - \frac{R_d}{c_p} \frac{\theta}{p} \Rightarrow T \frac{d \ln \theta}{dp} = \frac{dT}{dp} - \frac{TR_d}{pc_p}$$

  - By defining $S_p$ as

$$S_p = \frac{TR_d}{pc_p} - \frac{dT}{dp} = -\frac{T}{\theta} \frac{d\theta}{dp} \quad\Rightarrow\quad \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} - S_p \omega = \frac{\dot{Q}}{c_p}$$



# Conservation of Energy

- **Physical Meaning of $S_p$**
  - $S_p = -T d\ln\theta / dp$ is the equivalent, in isobaric coordinates, of the frequency of Brunt-Vaïsala $N^2 = g d\ln\theta_0 / dz$ in geometric coordinates
  - Thus $S_p$ represents the static stability parameter in isobaric coordinates

$$\frac{T}{\theta} \frac{\partial \theta}{\partial z} = \gamma_a - \gamma_e \quad\Rightarrow\quad \frac{T}{\theta} \frac{\partial \theta}{\partial p} = -\frac{\gamma_a - \gamma_e}{\rho g} \quad\Rightarrow\quad S_p = -\frac{T}{\theta} \frac{\partial \theta}{\partial p} = \frac{\gamma_a - \gamma_e}{\rho g}$$

  - $S_p > 0$ if $\gamma_e < \gamma_a$; as $\rho$ decreases exponentially with the height, $S_p$ increases very rapidly with $z$
  - This is a disadvantage for the definition of stability criterium
  - Finally, this equation allows to diagnose $\omega$:
    - by determining the advection of temperature from the thermal wind and measuring the local variation of $T$, it is possible to infer the $\omega$ value
    - $\omega$ is normally difficult to measure experimentally

# $\omega$ vs. w

- **$\omega$ vs. w**
  - By neglecting the diabatic heating: $\omega = S_p^{-1}\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right)$
  - Since $\omega$ has been determined, it can be evaluated w
  - Let's start by the definition of $\omega$ :

$$\omega = \frac{dp}{dt} = \frac{\partial p}{\partial t} + \mathbf{v} \bullet \nabla p + w\frac{\partial p}{\partial z} = \frac{\partial p}{\partial t} + \mathbf{v}' \bullet \nabla p - \rho g w$$

  where wind has been splited using geostrophic value $\mathbf{v} = \mathbf{v}_g + \mathbf{v}'$ with $|\mathbf{v}_g| \gg |\mathbf{v}'|$ ($\mathbf{v}_g \cdot \nabla p = 0$ being $\mathbf{v}_g \propto \nabla p$) and hydrostatic law has been used
  - In the last formula, a scale analysis show that
    - $\partial p/\partial t \cong 10$ hPa day$^{-1}$
    - $\mathbf{v}' \cdot \nabla p \cong 1$ hPa day$^{-1}$
    - $\rho g w \cong 100$ hPa day$^{-1}$
  - Thus, keeping the most important term:

$$\omega \approx -\rho g w$$

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